ON THE BOREL TRANSFORM OF AN ENTIRE FUNCTION OF EXponential TYPE

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ denote the Borel transform of the entire function defined by the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$ 

Then let

$$f_{L_1}(z) = \sum_{n=0}^{\infty} (n!) a_n z^n,$$

where $z, z_1 = 1$; similarly, let

$$f_{L_k}(z_k) = \sum_{n=0}^{\infty} (n!) a_n z_k^n,$$

where $z_{k-1}, z_k = 1$. Then all the $f_{L_k}(z_k)$ are entire functions on the $x$-plane or on the $(l/z)$-plane, according to the parity of $k$. A necessary and sufficient condition for $f_{L_k}(z_k)$ to be of finite order has been obtained, as well as some relations between the orders, lower orders, types and $\lambda$-types of $f(z)$ and $f_{L_k}(z_k)$.

1. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order $\rho$ and lower order $\lambda$. It is said to be of exponential type if it is of growth $(1, T)$ i.e., of order $\rho \leq 1$ and if $\rho = 1$, then its type is at the most equal to $T(T < \infty)$. Borel first showed (11, p. 73) that

$$(1.1) f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function of order 1 and type $\sigma$ if

$$(1.2) L. f(z) = \sum_{n=0}^{\infty} (n!) a_n z^n$$

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is convergent for $|z| \geq \sigma$. $L f(z)$, as usual, denotes the Borel transform of $f(z)$. We apply the transformation $z = \frac{1}{z_1}$ to (1.2) and denote

$$\frac{1}{z_1} L f \left( \frac{1}{z_1} \right)$$

by $f_L(z)$. Thus, we get

$$f_L(z) = \sum_{n=0}^{\infty} a_n (n!) z^n$$

where $f(z)$ and $L f(z)$ are in the same plane while $f_L(z)$ is in the new plane which we denote by $Z$-plane. If the order $\varrho$ of $f(z)$ is less than 1, then we show in this paper that $f_L(z)$ is also an entire function in the $Z$-plane. This is generalised by applying the Borel transform and inversion transform repeatedly. Thus, if $f_L(z)$, be of order $\varrho(0 < \varrho < 1)$ the application of the Borel transform and the inversion transform $z_i = \frac{1}{z_2}$ to (1.3) yields

$$f_L(z) = \sum_{n=0}^{\infty} a_n (n!) z^n$$

which will be an entire function in the $Z$-plane. Repeating the argument $k$ times, we can write

$$f_L(z) = \sum_{n=0}^{\infty} a_n (n!) k z^n$$

Evidently, if $k$ is an odd integer the function $f_L(z)$ will be in the $Z$-plane, while if $k$ is even, it will be in the $Z$-plane. It will follow that if $f_L(z)$, be of order $\varrho(k-1)$ an entire function of order $\varrho(k-1) < 1$ in one of these planes then $f_L(z)$ is an entire function in the other plane.

In this paper I have investigated a necessary and sufficient condition under which $f_L(z)$ is an entire function of finite order in one of these planes. A number of relations between orders, lower orders, types, and $\lambda$-types of $f(z)$ and $f_L(z)$ have also been obtained. The results are given in the form of theorems with remarks.

2. Theorem 1. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function of order $\varrho$, then

$$f_L(z_k) = \sum_{n=0}^{\infty} a_n (n!) k z^n$$

is an entire function of finite order in one of the planes, if and only if, $k \varrho < 1$.

Proof. We have
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\[
\frac{\log |a_n (n!)^k|^{-1}}{n \log n} \sim \frac{\log |a_n|^{-1}}{n \log n} - \frac{k \log (n!)}{n \log n}.
\]

But from STIRLING’s formula

\[n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}.
\]

Therefore,

\[
\frac{\log |a_n (n!)^k|^{-1}}{n \log n} \sim \frac{\log |a_n|^{-1}}{n \log n} - \frac{k (n+\frac{1}{2}) \log n}{n \log n} + O\left(\frac{1}{\log n}\right)
\]

\[= \frac{\log |a_n|^{-1}}{n \log n} - k + O(1).
\]

Since ([1], p. 9)

\[(2.1)\]

\[\lim \inf_{n \to \infty} \frac{\log |a_n|^{-1}}{n \log n} = \frac{1}{q},
\]

by making use of (2.1) in the above expression we get

\[(2.1)^\prime\]

\[\lim \inf_{n \to \infty} \frac{\log |a_n (n!)^k|^{-1}}{n \log n} = \frac{1}{q} - k.
\]

This leads us to the conclusion that \(f_{L_k}(z_k)\) is of finite order if \(k \varphi < 1\).

Conversely, let \(f(z)\) be of order \(\varphi\) (where \(k \varphi < 1\)); then, from (2.1)

\[\lim \sup_{n \to \infty} \frac{n \log n}{\log |a_n|^{-1}} = \varphi.
\]

Hence for any \(\varepsilon > 0\), we can find a number \(N(\varepsilon)\) such that

\[\frac{n \log n}{\log |a_n|^{-1}} < (\varphi + \varepsilon) \text{ for all } n > N(\varepsilon)
\]

or

\[|a_n| < n^{\frac{n}{\varphi + \varepsilon}}
\]

or

\[|(n!)^k a_n|^{1/n} < n^{\frac{1}{\varphi + \varepsilon}}.
\]

Therefore

\[\lim_{n \to \infty} |(n!)^k a_n|^{1/n} = 0
\]

and hence the theorem.

Theorem 2. Let

\[f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

be of order \(\varphi\) \((m \varphi < 1)\) and lower order \(\lambda\) and let \(\varphi_k\) and \(\lambda_k\) denote respectively the order and lower order of
\[ f_{L_k}(z_k), \quad (k = 1, 2, \ldots, m), \]
then
\[ \varrho_k = \frac{\varrho}{1 - k \varrho}. \]

If further,
\[ \left| \frac{a_n}{(n+1)k a_{n+1}} \right|, \quad (k = 1, 2, \ldots, m) \]
forms a non-decreasing function of \( n \) for \( n > n_0 \), then
\[ \lambda_k = \frac{\lambda}{1 - k \lambda}. \]

**Proof.** (2.2) follows from (2.1)'). Further, since
\[ \left| \frac{a_n}{(n+1)k a_{n+1}} \right|, \quad (k = 1, 2, \ldots, m) \]
forms a non-decreasing function of \( n \) for \( n > n_0 \), in view of the fact ([3], p. 1047), that
\[ \log n \]
we get
\[ \lim_{n \to \infty} \sup \log \left| \frac{a_n (n+1)^k}{n \log n} \right| = \frac{1}{\lambda_k}. \]

Using Stirling's formula for \( n! \), we easily get
\[ \lambda_k = \frac{\lambda}{1 - k \lambda}. \]

**Applications.** Let
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]
be of order \( \varrho \), \( (m \varrho < 1) \) lower order \( \lambda \) and
\[ f_{L_k}(z_k) = \sum_{n=0}^{\infty} a_n (n+1)^k z_k^n \]
be of order \( \varrho_k \) and lower order \( \lambda_k \) respectively and satisfy the hypothesis of theorem 2: then the direct consequences of (2.2) and (2.3) are
\[ \varrho < \varrho_1 < \varrho_2 < \ldots < \varrho_{m-1} < \varrho_m \quad \text{if and only if} \quad \varrho \neq 0, \]
\[ \lambda < \lambda_1 < \lambda_2 < \ldots < \lambda_{m-1} < \lambda_m \quad \text{if and only if} \quad \lambda \neq 0. \]
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(2.7) \( q = 0 \) if and only if \( \varphi_k = 0 \) for any \( k, \ 1 \leq k \leq m \),

(2.8) \( \lambda = 0 \) if and only if \( \lambda_k = 0 \) for any \( k, \ 1 \leq k \leq m \),

(2.9) \( q = \lambda \) if and only if \( \varphi_k = \lambda_k \) for any \( k, \ 1 \leq k \leq m \),

i.e. if \( f(z) \) is of regular growth then so is \( f_{L_k}(z_k) \) and vice versa.

Theorem 3. Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

be an entire function of order \( q, (0 < q < 1) \) and type \( T \). Then

(2.10) \[ T_k = \frac{\varphi_k}{\varepsilon_k (q T)^{\theta}} \text{ for } k = 1, 2, \ldots, m \]

where \( \varphi_k \) and \( T_k \) are the order and type, respectively, of \( f_{L_k}(z_k) \).

Proof. Let

\[ \psi(n) = \frac{n}{\varepsilon_k} |a_n(n)^{\theta}|^{\frac{\varphi_k}{n}}. \]

Then

\[ \log \psi(n) = \log \frac{1}{\varepsilon_k} + \log n + \frac{k \varphi_k}{n} \log (n!) + \frac{\varphi_k}{n} \log |a_n|. \]

Since, from (2.2),

\[ \varphi_k = \frac{q}{1 - k q} \text{ and } n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}, \]

by making use of these in the above expression, we get

\[ \log \psi(n) \sim \log \frac{1}{\varepsilon_k} + \frac{\varphi_k}{\varepsilon_k} \log n + \frac{\varphi_k}{\varepsilon_k} \log |a_n|^{q/n}. \]

Now, proceeding to limit, we get

\[ T_k = \frac{1}{\varepsilon_k} (q T)^{\theta} \]

in view of the fact ([1], p. 11) that
and hence the theorem is proved.

Let \( f(z) \) be of order \( \rho \) and lower order \( \lambda, (0 \leq \lambda < \rho < \infty) \), then \([1]\),

\[
\lim_{n \to \infty} \sup_{e^{\theta_0}} \frac{1}{n} \left| a_n \right|^{\rho/n} = T
\]

Further, let

\[
\lim \inf_{y \to \infty} \frac{\log M(y)}{y^\rho} = 0.
\]

We call \( t_k \) the \( \lambda \)-type of \( f(z) \). It has also been shown \([1]\) that, if \( (0 < \lambda < \infty) \) and

\[
\lim \inf_{n \to \infty} \frac{n}{e^\lambda} \left| a_n \right|^{1/n} = t_k.
\]

The following theorem can be proved on the lines of the proof of theorem 3.

**Theorem 4.** Let

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

be an entire function of order \( \rho \), lower order \( \lambda (0 < \lambda < \infty) \) and \( \lambda \)-type \( t_k \). If

\[
\left| \frac{a_n}{(n+1)^k a_{n+1}} \right|, \quad (k = 1, 2, \ldots, m)
\]

forms a non-decreasing function of \( n \) for \( n > n_k \), then

\[
t_k^k = \frac{1}{t_k^k} \left( \lambda_k t_k \right)^{\frac{1}{k}}
\]

for

\[
k = 1, 2, \ldots, m.
\]

where \( \lambda_k \) and \( t_k \) are lower order and \( \lambda \)-type of \( f_{k}(z) \).

**Remark.** If \( f(z) \) is of regular growth i.e. \( q = \lambda \) then (2.15) is the relation involving the lower types of \( f_{k}(z) \) and \( f(z) \). Otherwise, if \( q \neq \lambda \) it follows from (2.12) that lower type of \( f(z) \) is zero.

Here we give a few applications of theorems 2, 3 and 4.
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(i) The relation, (2.2), (2.3), (2.10) and (2.15) are recurrence relations. Hence knowing the order and type of any one function out of the $m+1$ functions, one can find out the order and type of any of the other $m$ functions. The same is true for the lower order and $\lambda$-type.

(ii) 

$$(\theta_k T_k)^{1/\lambda_k} \text{ and } (\lambda_k T_{\lambda_k})^{1/\lambda_k}$$

are invariant quantities for

$$k = 1, 2, \ldots, m.$$ 

(iii) If

$$\left| \frac{a_n}{(n+1)^k a_{n+1}} \right|, \quad (k = 1, 2, \ldots, m)$$

forms a non-decreasing function of $n$ for $n > n_0$, then $f_{(\lambda_k)}(\gamma_k)$ is of perfectly regular growth if and only if $f(\gamma)$ is of perfectly regular growth.

(iv) If $g \neq \lambda$ then $t_k = 0$ for $k = 1, 2, \ldots, m$, $t_k$ is the lower type of $f_{(\lambda_k)}(\gamma_k)$.

(v) If $m > 0$, then $f(\gamma)$ and $f_{(\lambda_k)}(\gamma_k)$ each have an infinity of zeroes in their respective planes.

(vi) If one considers $(0, \varrho), (1, \varrho), \ldots, (m, \varrho_m)$ as points in the cartesian plane then all lie on the curve

$$y = \frac{\varrho}{1-\gamma \varrho}.$$

Theorem 5. Let

$$f(\gamma) := \sum_{n=0}^{\infty} a_n \gamma^n$$

be an entire function of order, $\varrho$ lower order $\lambda$ ($0 < \lambda$, $m \varrho < 1$), type $T(T >)$ and $\lambda$-type $\lambda, (\lambda > 0)$, then

(2.16)

$$\frac{a_m T_m}{a T} = \prod_{k=1}^{m} (\theta_{k-1} T_{\lambda_{k-1}})^{\lambda_k}$$

and

(2.17)

$$a_m - \varrho = \sum_{k=1}^{m} \theta_{k-1} \theta_k \varrho.$$ 

If further, $\left| \frac{a_n}{a_{n+1}} \right|$ forms a non-decreasing function of $n$ for $n > n_0$, then

(2.18)

$$\frac{\lambda_m T_{\lambda m}}{\lambda T_{\lambda}} = \prod_{k=1}^{m} (\lambda_{k-1} T_{\lambda_{k-1}})^{\lambda_k}$$

and
(2.19) \[ \lambda_m - \lambda = \sum_{k=1}^{m} \lambda_{k-1} \gamma_k \]

where for \( k = 1, \) \( \varrho_0 = \varrho, \) \( \lambda_0 = \lambda, \) \( T_0 = T, \) \( t_{\lambda_0} = t_{\lambda} \) and \( \varrho_k, \lambda_k, T_k \) and \( t_{\lambda_k} \) are the same as in theorems 2, 3 and 4.

Proof. Let us consider the entire functions \( f_{L_{k-1}}(z_{k-1}) \) and \( f_{L_k}(z_k) \) then, on the basis of theorem 3, we can see that

(2.20) \[ T_k = \frac{1}{\varrho_m} (\varrho_{k-1} T_{k-1})^{\varrho_k}. \]

Putting \( k = 1, 2, \ldots, m \) in (2.20) and then multiplying the \( m \) equations thus obtained, we get

(2.21) \[ T_1 T_2 \ldots T_m = \frac{1}{\varrho_1 \varrho_2 \ldots \varrho_m} (\varrho T)^{\varrho_1} \ldots (\varrho_{m-1} T_{m-1})^{\varrho_m}. \]

Again, considering \( f_{L_{k-1}}(z_{k-1}) \) in place of \( f(z) \) in theorem 2, then (2.2) reduces to

(2.22) \[ \varrho_k = \frac{\varrho_{k-1}}{1 - \varrho_{k-1}} \]

\( i.e.\) \( \varrho_k - \varrho_{k-1} = \varrho_k \varrho_{k-1}. \)

Making use of (2.22) in (2.21), we can easily see that

\[ \frac{\varrho_m T_m}{\varrho T} = \prod_{k=1}^{m} (\varrho_{k-1} T_{k-1})^{\varrho_k} \]

which is (2.16).

In (2.22), putting \( k = 1, 2, \ldots, m \) and then adding all the equations thus obtained, we get

\[ \sum_{k=1}^{m} (\varrho_k - \varrho_{k-1}) = \sum_{k=1}^{m} (\varrho_k \varrho_{k-1}) \]

or

\[ \varrho_m - \varrho = \sum_{k=1}^{m} \varrho_k \varrho_{k-1} \]

which is (2.17). Similarly, we can prove (2.18) and (2.19).

3. Now, we obtain relations between the maximum moduli of \( f(z) \) and \( f_{L_k}(z_k) \) and also between their maximum terms and their ranks. We denote by \( M(\gamma) \) the maximum modulus of \( f(z) \) for \( |z| = \gamma, \) and by \( M(\gamma, f_{L_k}) \) the maximum modulus of \( f_{L_k}(z_k), \) taking \( |z_k| = |z| = \gamma. \) When \( k \) is even we have \( z = z_k \) but when \( k \) is odd we have \( z = 1/z_k. \)
In the latter case, having chosen a value $\gamma$ for $|z|$, we look for the point in the $Z$-plane such that $|zk| = \gamma$. Corresponding to this point $zk$ the point in the $z$-plane will have the modulus $1/\gamma$. Similar remarks apply also for the maximum terms and for their ranks in the following theorems.

Theorem 6. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order $\rho (0 \leq \rho < 1)$ and lower order $\lambda$. If

$$\left| \frac{a_n}{(n+1)^m a_{n+1}} \right|$$

forms a non-decreasing function of for $n > n_k$, then for any $\varepsilon > 0$.

$$\frac{1-\rho+\varepsilon}{1-k\rho} \left\{ \log M(\gamma) \right\}^{1-k\rho} < \log M(\gamma, f_{L_k}) < \frac{\varepsilon-\lambda+\varepsilon}{1-k\rho} \left\{ \log M(\gamma) \right\}^{1-k\rho}$$

for $\gamma < \gamma_k(\varepsilon)$ and $k = 1, 2, \ldots, m$.

Proof. It is known (\cite{1}, p. 9) that

$$\lim_{\gamma \to \infty} \sup \log \log M(\gamma) = e \lambda.$$

Therefore, for any $\varepsilon' > 0$, we can find a positive number $\gamma(\varepsilon')$ such that

$$\gamma^{\lambda-\varepsilon'} < \log M(\gamma) < \gamma^{\rho+\varepsilon'}.$$ (3.1)

Similarly, for the integral function $f_{L_k}(zk)$, we have

$$\gamma^{\lambda-k\rho+\varepsilon_k} < \log M(\gamma, f_{L_k}) < \gamma^{\rho-k\rho+\varepsilon_k}$$

for $\gamma > \gamma_k(\varepsilon_k)$

or

$$\gamma^{\lambda-\rho-k\rho} \left\{ \log M(\gamma) \right\}^{1-k\rho} < \log M(\gamma, f_{L_k}) < \gamma^{\rho-k\rho} \left\{ \log M(\gamma) \right\}^{1-k\rho}$$

where

$$\varepsilon_k \geq \left( \varepsilon_k + \frac{1-k\rho}{\varepsilon'} \right).$$

Making use of (3.1) in the above inequality, we get

$$\gamma^{\lambda-k\rho} \left\{ \log M(\gamma) \right\}^{1-k\rho} < \log M(\gamma, f_{L_k}) < \gamma^{\rho-k\rho} \left\{ \log M(\gamma) \right\}^{1-k\rho}.$$
where

$$\gamma > \gamma_0 (e) = \max_{1 \leq k \leq m} \{ \gamma_0 (e'), \gamma_k (e_k) \}$$

and hence the theorem.

**Theorem 7.** Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order \( e \) \((0 \leq e < 1)\), lower order \( k \) and let

$$v (\gamma, f), v (\gamma, f_{L_k}), v (\gamma, f^{(s)}) \text{ and } v (\gamma, f^{(s)}_{L_k})$$

denote respectively the ranks of the maximum terms of

$$f(z), f_{L_k} (z)$$

and their \( s \)-th derivatives

$$f^{(s)} (z) \text{ and } f^{(s)}_{L_k} (z_k).$$

If

$$\left| \frac{a_n}{(n+1)^m a_{n+1}} \right|$$

forms a non-decreasing function of \( n \) for \( n > n_0 \), then for any \( e > 0 \), we have

$$\frac{\lambda - e - \varepsilon}{1 - k \varepsilon} + \frac{1}{(1 - k \varepsilon) s \log \gamma} \int_{\gamma_0}^{\gamma} \frac{\varphi (x, s)}{x} \, dx < \frac{1}{s \log \gamma} \int_{\gamma_0}^{\gamma} \frac{\varphi_k (x, s)}{x} \, dx$$

for \( \gamma > \gamma_0 \) and \( k = 1, 2, \ldots, m \), where

$$\varphi (\gamma, s) = v (\gamma, f^{(s)}) - v (\gamma, f),$$

$$\varphi_k (\gamma, s) = v (\gamma, f^{(s)}_{L_k}) - v (\gamma, f^{(s)}_{L_k}).$$

**Proof.** It is known ([4], p. 276) that

$$\lim_{\gamma \to \infty} \sup_{\gamma} \inf_{x} \frac{1}{s \log \gamma} \int_{\gamma_0}^{\gamma} \frac{\varphi (x, s)}{x} \, dx = \frac{e}{\lambda}.$$

Therefore, proceeding on the same lines as in theorem 6 we easily obtain the result.
Theorem 5. Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

be an entire function of order \( \sigma (0 \leq m \sigma < 1) \), lower order \( \lambda \) and let

\[ \mu (\gamma, f), \mu (\gamma, f_{L_k}), \mu (\gamma, f^{(e)}), \mu (\gamma, f^{(e)}_{L_k}) \]

denote respectively the maximum terms of

\[ f(z), f_{L_k}(z), f^{(e)}(z) \text{ and } f^{(e)}_{L_k}(z). \]

If

\[ \frac{a_n}{(n+1)^m a_{n+1}} \]

is a non-decreasing function of \( n \) for \( n > n_0 \), then for any \( \varepsilon > 0 \),

\[ \left[ \frac{\mu (\gamma, f^{(e)}_{L_k})}{\mu (\gamma, f_{L_k})} \right]^{1/(1-k \rho)} < \gamma \left[ \frac{\mu (\gamma, f^{(e)})}{\mu (\gamma, f)} \right] < \left[ \frac{\mu (\gamma, f^{(e)})}{\mu (\gamma, f)} \right]^{1/(1-k \rho)} \]

for

\[ \gamma > \gamma_0 (\varepsilon) \text{ and } k = 1, 2, \ldots, m. \]

Proof. It is known ([5], p. 107) that

\[ \lim_{\gamma \to 0} \sup \frac{\log \gamma \left\{ \frac{\mu (\gamma, f^{(e)})}{\mu (\gamma, f)} \right\}^{1/e}}{\log \gamma} = e. \]

Again, proceeding on the same lines as in theorem 6 we get the result (').

REFERENCE


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ÖZET

\( Lf(z) = \sum_{n=0}^{\infty} (n!) a_n z^n, f(z) = \sum_{n=0}^{\infty} a_n z^n \) seri açılımı ile verilen tam fonksiyonun 

Bönen dönüştürülmişini gösterin. Bu takdirde, \( z \cdot z_1 = 1 \) olmak üzere,

\[ f_{L_1}(z_1) = \sum_{n=0}^{\infty} (n!) a_n z_1^n \]

vaz edilsin ve buna benzer tarzda, \( z_k-1 \cdot z_k = 1 \) olmak üzere,

\[ f_{L_k}(z_k) = \sum_{n=0}^{\infty} (n!)^k a_n z_k^n \]

tanımlansın, \( f_{L_k}(z_k) \) fonksiyonları, \( k \) sayısıının çift veya tek olmasına göre, \( z \) veya 

\( 1/z \) düzleminde tam fonksiyonlardır. Bu yüzden bu fonksiyonların sonlu mertebeden olmaları için bir gerek ve yeter şart oldu edilmiş, ayrıca \( f_{L_k}(z_k) \) ile \( f(z) \) fonksiyon-

ların mertebeilleri, alt mertebeilleri, tipleri ve \( \lambda \)-tipleri arasında bazı bağıntılar 

işpat edilmiştir.