Dirac Equation in a 5-dimensional Kaluza-Klein Theory

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Abstract. Dirac equation is discussed in 5-dimensional space time having topology $M^4 \times T^1$, where $M^4$ and $T^1$ both are curved. It is shown that 4-dimensional fermion can be obtained from 5-dimensional fermion, as a result of compactification of extra dimension. It is found that the realistic 4-dimensional fermions are possible in higher modes earlier than those in lower modes during the course of expansion of 4-dimensional universe. 4-dimensional Dirac equation, obtained from 5-dimensional Dirac equation after compactification, is solved for an arbitrary modes for super-heavy as well as light (realistic) fermions. Time-dependence of polarization vector and magnetization density, as a result of Gordon-decomposition of the current vector for 4-dimensional spin-$\frac{1}{2}$ field (with arbitrary mode), is exhibited.

Keywords: Fermions, Dirac equation, Kaluza-Klein theory, energy- momentum tensor.

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1 Introduction

In the context of unification of gravity with gauge interactions, Kaluza-Klein type theories [1-4] are good candidates. In these theories, the spacetime is supposed to have the topology $M^4 \times T^{n-4}$ ($n$ is the total number of dimension), where $M^4$ denotes the usual para-compact four-dim. spacetime (flat or curved) and $T^{n-4}$ is the extra (n-4)-dim. compact manifold. The observed universe is 4-dimensional, hence it is supposed that the extra-manifold due to gauge interactions might be compact and very small in size so that these are not observed today. Physically, it is very much appealing to think that $T^{n-4}$ manifold is curved due to its compact nature[5]. The action for higher-dim. gravity is

$$s = \int d^n x \sqrt{-g} R$$

where $g = det \ g_{\mu\nu}$ ($g_{\mu\nu}$ is the metric tensor for the n-dimensional space time) and $R$ is the Ricci curvature scalar for $M^4 \times T^{n-4}$. n-dimensional Einstein’s equations derived from the above action yields that $T^{n-4}$ can be curved (1) when $M^4$ is also curved or (2) action contains some lagrangian for matter field also so that energy-momentum tensor is non-vanishing[6]. This discussion shows that if $M^4$ is curved, the

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extra-dim. manifold will definitely be curved. Motivated by this idea, the line-element
for 5-dim. space time is taken as

\[ ds^2 = dt^2 - a^2(t)[dx_1^2 + dx_2^2 + dx_3^2] - A^2(t)dy^2 \]  

(1.1)

Using the metric tensor from (1.1) in the Einstein’s field equations[7] derived from the
above action, it is interesting to see that

\[ aA = 1 \]  

(1.2)

Thus (1.1) deserves a model in which one spatial dimension shrinks with time while
the other three expand. Earlier, Chodos and Detweiler[8] have considered this type of
spacetime.

In 4-dim. spacetime, Dirac equation has been solved and discussed by many authors[9].
The aim of present paper is to discuss and solve Dirac equation for spin-$\frac{1}{2}$ field \( \Psi_5 \) in
5-dim. spacetime (1.1) for different time-function \( a(t) \).

The degrees of freedom for a spin-$\frac{1}{2}$ field \( \Psi \) in \( n \) dimensions[10] is given by \( 2^\alpha \) where
\( \alpha = \frac{n}{2} \) (if \( n \) is even) and \( \alpha = \frac{(n-1)}{2} \) (if \( n \) is odd). The dimension of the space-time, here,
is 5, so degrees of freedom for \( \Psi \) is 4. The flat space Dirac matrices in 5-dim. will be
\( \tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_5 \) where \( \tilde{\gamma}_5 = \tilde{\gamma}_0 \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \) (other \( \gamma \)-
matrices are the usual standard matrices
for 4-dim.,)[11]. Now for Weyl’s transformation a matrix \( \tilde{\gamma} = \tilde{\gamma}_0 \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \tilde{\gamma}_5 = 1 \) can be
defined. So operation of chiral projection operators \( \frac{1}{2}(1 \pm \tilde{\gamma}) \) on the Dirac equation

\[ i\tilde{\gamma}^\mu \partial_\mu \Psi_5 + m_5 \Psi_5 = 0 \]  

(1.3)

shows that chiral fermions are not possible in 5-dim.[10]. Hence \( m_5 \), the mass of \( \Psi_5 \) is not
zero. This is true for fermions in curved spacetime also which obey the Dirac equation

\[ i\gamma^\mu D_\mu \Psi_5 + m_5 \Psi_5 = 0 \]  

(1.4)

where \( \gamma^\mu \) are curved space Dirac-matrices and \( D_\mu \) denotes covariant derivatives in curved
spacetime (1.1).

The paper is planned as follows: Section 2 contains 5-dim. Dirac Equation in the space-
time (1.1) and a brief discussion on the effective mass of 4-dim. fermion produced by
5-dim. fermion is given. Section 3 contains some explicit examples of solutions for 5-dim.
Dirac Equation. In the last section, Gordon-decomposition[12] of 4-dim. \(\Psi_4\), obtained after compactification of 5-dim spacetime, has been discussed. \(\hbar = c = 1\) is used as fundamental unit where \(\hbar\) and \(c\) carry their usual meaning. Here indices \(a, \mu \cdots = 0, 1, 2, 3, 5\).

2. Dirac Equation for \(\Psi_5\)

The vierbein \(h^\mu_a\) on the manifold \(M^4 \times T^1\) (\(T^1\) is circle) is defined as

\[
h^\mu_a h^\nu_b g_{\mu\nu} = \eta_{ab} \tag{2.1}
\]

where \((\mu, \nu)\) are curved space indices, \((a, b)\) are flat space indices, \(g_{\mu\nu}\) is the curved space metric tensor and \(\eta_{ab}\) is the Minkowskian metric. So, in the space time (1.1)

\[
h^0_0 = 1, h^1_1 = h^2_2 = h^3_3 = a^{-1}(t), h^5_5 = A^{-1}(t) = a(t) \tag{2.2}
\]

The operator \(D_\mu\) in (1.4) is defined as [13]

\[
D_\mu = \partial_\mu - \Gamma_\mu \tag{2.3}
\]

where \(\Gamma_\mu\) is the spin coefficient or Ricci rotation coefficient given as

\[
\Gamma_\mu = \frac{1}{4} \left( \partial_\mu h^\rho_a + \left\{ \rho_\mu \right\} h^\rho_a \right) h^\nu_b g_{\rho\nu} \tilde{\gamma}^a \tilde{\gamma}^b \tag{2.4}
\]

where \(\left\{ \rho_\mu \right\}\) is the affine-connection. 
\(\tilde{\gamma}^a\) are flat space Dirac matrices satisfying the anti-commutation rule

\[
\left\{ \tilde{\gamma}^a, \tilde{\gamma}^b \right\} = 2\eta^{ab}. \tag{2.5}
\]

Curved space Dirac matrices \(\gamma^\mu\) satisfy

\[
\left\{ \gamma^\mu, \gamma^\nu \right\} = 2g^{\mu\nu} \tag{2.6}
\]

\(\gamma^\mu\) is related to \(\tilde{\gamma}^a\) through vierbein as

\[
\gamma^\mu = h^\mu_a \tilde{\gamma}^a \tag{2.7}
\]

So, Dirac equation in (1.1) is written as

\[
i \left[ \tilde{\gamma}^0 \left( \partial_0 + \frac{9}{4} \frac{\dot{a}}{a} \right) + a^{-1} (\tilde{\gamma}^1 \partial_1 + \tilde{\gamma}^2 \partial_2 + \tilde{\gamma}^3 \partial_3) + a \tilde{\gamma}^5 \partial_5 \right] \Psi_5 - m_5 \Psi_5 = 0, (\dot{a} = \partial_0 a) \tag{2.8}
\]
The internal space is compactified, so

\[ \Psi_5(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(y) \Psi_4^{(n)}(x) \]  

(2.9)

where \( \Psi_4^{(n)}(x) \) are 4-dim. spinor fields in nth mode and \( \phi_n(y) \) are harmonic functions which obey the equation[4]

\[ i\tilde{\gamma}^5 \partial_5 \phi_n(y) = \frac{n}{R} \phi_n(y) \psi_4^{(n)} \]  

(2.10)

\( R \) is the compactification scale. \( \phi_n \) is normalised as

\[ \phi_n \phi_m = \delta_{nm}. \]

Connecting (2.8) and (2.10)

\[ \left[ i\{\tilde{\gamma}^0 (\partial_0 + \frac{9\dot{a}}{4a}) + a^{-1}(\tilde{\gamma}^1 \partial_1 + \tilde{\gamma}^2 \partial_2 + \tilde{\gamma}^3 \partial_3)\} - (m_5 - \frac{an}{R}) \right] \Psi_4^n = 0 \]  

(2.11)

(2.11) shows that effective mass for \( \Psi_4^n \) is

\[ M = m_5 - \frac{an}{R} = m_5 - anM_c \]  

(2.12)

where \( R^{-1} = M_c \) is the compactification mass.

The Extra dimension \( y \) which is assumed to be a circle of radius \( R \), has the range

\[ 0 \leq y \leq 2\pi R \]  

(2.13)

The distance around the extra dimension is given by [13]

\[ \delta_5 = \int_0^{2\pi R} dy \sqrt{-g_{55}} = \frac{2\pi R}{a(t)} \]  

(2.14)

which is time-dependent and decreases as \( a(t) \) increases.

At \( t = t_c \) (compactification time), it is assumed that

\[ \frac{2\pi R}{a(t_c)} \sim L_P \quad \text{(PlankLength)} \]  

(2.15)

For realistic fermions, \( M \simeq 0 \). So, as a particular time \( t = t_P \)

\[ m_5 \simeq \frac{na(t_P)}{R} \]  

(2.16)
Connecting (2.15) and (2.16)

\[ m_5 \simeq \frac{2\pi n a(t_p) M_P}{a(t_c)} \]  

where \( M_P \) is the Planck mass and is equal to \((L_P)^{-1}\)

Hence from (2.17)

\[ n \simeq \frac{m_5 a(t_c)}{2\pi a(t_p) M_P} \leq \frac{m_5}{2\pi M_P} \]  

as \( a(t_p) \gtrsim a(t_c) \) in the expanding 4-dim. universe. So (2.18) puts a constraint on modes \( n \). Also (2.18) states that if one gets realistic 4-dim. fermion from 5-dim fermion as a result of compactification of the extra dimension, number of physically allowed modes depends on \( m_5 \). For example, \( n = 0, 1 \) if \( m_5 = 2\pi M_p \); \( n = 0, 1, 2 \) if \( m_5 = 4\pi M_p \); \( n = 0, 1, 2, ..., (r-1) \), \( r \) if \( m_5 = 2\pi r M_p \) where \( r \) is positive integer. From (2.16)

\[ a(t_p) \simeq \frac{m_5 R}{n} \]  

So, one gets

\[ a(t_p) \simeq \frac{2\pi r M_p R}{n} \]  

where \( n = 0, 1, 2, ...(r-1), r \). Now it is interesting to see that if \( a(t_{p1}) = a_1 \) when \( n = r \) and \( a(t_{p2}) = a_2 \) when \( n = r-1, a_2 > a_1 \). It means that realistic fermions may be obtained in higher modes earlier than those in lower modes during the course of expansion of the observable 4-dim. universe, as \( t_{p1} < t_{p2} \).

3. Solution of Dirac Equation

Substituting \( \Psi \) defined as

\[ \psi_4^{(n)} = a^{-\frac{9}{4}} \Psi \]  

(2.11) is re-written as

\[ \left[ (\gamma^0 \partial_\tau + \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3) + ia(m_5 - anM_c) \right] \Psi = 0 \]  

with

\[ \tau = \int^t d\tau' \frac{dt'}{a(t')} \]  

Writing

\[ \Psi = \sum_k \Psi_k = (2\pi)^{-\frac{3}{2}} \exp(ikx) \left[ f_i(k, \tau) \right] \]
One gets two coupled equations

\[
[\partial_{\tau} + ia(m_5 - anM_c)]f_I + i(k.\sigma)f_{II} = 0 \quad (3.5a)
\]

\[
[\partial_{\tau} - ia(m_5 - anM_c)]f_{II} + i(k.\sigma)f_I = 0 \quad (3.5b)
\]

where \(\sigma_1, \sigma_2, \sigma_3\) are pauli matrices.

From (3.5a)

\[
f_{II} = ik^{-2}(k.\sigma)[\partial_{\tau} + ia(m_5 - anM_c)]f_I \quad (3.6)
\]

Connecting (3.5b) and (3.6)

\[
f''_I + [k^2 + a^2(m_5 - anM_c)^2 + i\partial_{\tau}\{a(m_5 - anM_c)\}]f_I = 0 \quad (3.7)
\]

where prime (/) denotes differentiation with respect to \(\tau\).

The norm for \(\psi_5\) is defined at \(\tau=\)constant hypersurface as

\[
(\psi_{5ks}, \psi_{5k's'}) = \int_\tau \sqrt{|g_5|} d^3x dy \bar{\psi}_{5ks} \gamma^0 \psi_{5k's'} \quad (3.8)
\]

where \(g_5\) is the determinant of 5-dim. metric tensor.

Connecting (2.9) and (3.8)

\[
(\psi_{5ks}, \psi_{5k's'}) = \int_\tau \sqrt{|g_5|} d^3x dy \sum_n \bar{\psi}_{4ks}^n(x) \phi_n(x) \gamma^0 \sum_{n'} \phi_{n'}(y) \psi_{4k's'}^{n'}(x) \quad (3.9)
\]

which yields, on integration over extra dimension \(y\) having the range \(0 \leq y \leq 2\pi R\)

\[
(\psi_{5ks}, \psi_{5k's'}) = 2\pi R \int_\tau \sqrt{|g_5|} d^3x \sum_n \bar{\psi}_{4ks}^n \gamma^0 \psi_{4k's'}^{n} \quad (3.10)
\]

The normalization constants are determined using the prescription that in the flat space limit

\[
(\psi_{5ks}, \psi_{5k's'}) \longrightarrow \delta_{ss'}\delta^3(k - k') \quad (3.11)
\]

or,

\[
2\pi R \int_\tau \sqrt{|g_5|} d^3x \sum_n \bar{\psi}_{4ks}^n \gamma^0 \psi_{4k's'}^{n} \longrightarrow \delta_{ss'}\delta^3(k - k') \quad (3.12)
\]

Now, some explicit examples of solutions of Dirac equation for \(\psi^n_4\) utilising the above mentioned procedure of normalization are given as under for different \(a(t)\).

\[
3(A) \quad a(t) \simeq t^{\frac{1}{2}}
\]
From (3.3) 

\[ \tau = 2t^{\frac{1}{2}} \text{ and } a(t) = \frac{1}{2} \tau, \]  

so (3.7) is written as 

\[ f''_I + \left[ k^2 + \frac{\tau^2}{4} \left( m_5 - \frac{nM_c \tau}{2} \right)^2 + i \frac{\partial}{\partial \tau} \left\{ \tau \left( m_5 - \frac{nM_c \tau}{2} \right) \right\} \right] f_I = 0 \]  

(3.14) can be approximated in two different ways: 

**Case I:** When \( m_5 >> \frac{1}{2} nM_c \tau \) (3.14) is approximated as 

\[ f''_I + \left[ k^2 + \frac{im_5}{2} - \frac{inM_c \tau}{4} + \frac{m_5^2 \tau^2}{4} \right] f_I = 0 \]  

which has exact solution 

\[ f_I = \exp \left[ \pm \frac{nM_c \tau}{4m_5} \mp \frac{im_5 \tau^2}{4} \right] \left[ N_1 \ nF_1(a, 1 \frac{1}{2}, X) + \right. \]  

\[ N_2 e^{i(\pi r + 1) \tau} \left( \frac{nM_c}{2m_5} - \frac{im_5 \tau}{2} \right) \left( nF_1(1 \frac{1}{2} + a, 3 \frac{3}{2}, X) \right) \]  

(3.16) 

where 

\[ a = \pm \frac{i}{m_5} \left( \frac{n^2 M_c^2}{32m_5^4} + \frac{k^2}{2m_5^2} \right), \]  

\[ X = \pm \frac{i}{2m_5} \left( \frac{nM_c}{2m_5} - \frac{im_5 \tau}{2} \right)^2, \]  

\( nF_1(a, c, X) \) is confluent hypergeometric function and \( r = 0, 1, 2, \ldots \). 

Connecting (3.6) and (3.16) 

\[ f_{II} = \frac{i(k\sigma)}{k^2} \exp \left[ \pm \frac{nM_c \tau}{4m_5} \mp \frac{im_5 \tau^2}{4} \right] \times \]  

\[ \left[ N_3 \left\{ \left( \pm \frac{nM_c}{4m_5} \mp \frac{im_5}{2} \mp \frac{im_5}{2} \mp \frac{im_5 \tau^2}{4} \right) \right\} nF_1(a, 1 \frac{1}{2}, X) + 2aX' nF_1(1 + a, 3 \frac{3}{2}, X) \right\} + \right. \]  

\[ N_4 e^{i(\pi r + 1) \tau} \left( \frac{nM_c}{2m_5} - \frac{im_5 \tau}{2} \right) \left( \frac{im_5}{2} + \frac{im_5}{2} - \frac{im_5 \tau^2}{4} \right) nF_1(1 + a, 3 \frac{3}{2}, X) - \]  

\[ \left. i \frac{\sqrt{m_5}}{2} nF_1(1 \frac{1}{2} + a, 3 \frac{3}{2}, X) \right] + \frac{1 + 2a}{3} X' \frac{\frac{nM_c}{2m_5} - \frac{im_5 \tau}{2}}{\frac{\sqrt{2m_5}}{2}} nF_1(1 \frac{3}{2} + a, 3 \frac{5}{2}, X) \]  

(3.17) 

Corresponding to \( f_I \) and \( f_{II} \), \( \psi^n_{k41s} \) and \( \psi^n_{k41Is} \) are written as 

\[ \psi^n_{k41s} = (2\pi)^{-\frac{1}{2}} \left( \frac{r}{2} \right)^{-\frac{1}{2}} \exp[ikx \pm \frac{nM_c \tau}{4m_5} \mp \frac{im_5 \tau^2}{4}] \times \left[ N_1 u_s nF_1(a, \frac{1}{2}, X) + \right. \]  

\[ N_2 \bar{u}_s e^{i(\pi r + 1) \tau/4} \frac{\frac{nM_c}{2m_5} - \frac{im_5 \tau}{2}}{\frac{\sqrt{2m_5}}{2}} nF_1(1 \frac{1}{2} + a, 3 \frac{3}{2}, X) \]  

(3.18)
\[ \psi_{kM} = (2\pi)^{-\frac{3}{2}} \left( \frac{\tau}{2} - \frac{i(k, \sigma)}{k^2} \exp[ik.x \pm \frac{nM_c x}{4m_5} \tau \mp \frac{im_5 \tau^2}{4}] \right) \times \]

\[ [N_3 u_s \{ (\pm \frac{nM_c}{4m_5} + \frac{im_5}{2} \tau + \frac{int^2 M_c}{4} ) \}_1 F_1 (a, \frac{1}{2}, X) + 2aX' \}_1 F_1 (1 + a, \frac{3}{2}, X)] + \]

\[ N_4 \hat{u}_s e^{i(\theta + \pi)/4} \{ (\pm \frac{nM_c}{2m_5} - \frac{im_5}{2} \tau \}_1 F_1 (a, \frac{1}{2}, X) + 2aX' \}_1 F_1 (1 + a, \frac{3}{2}, X)] \]

\[ \times \left[ \sqrt{\frac{m_5}{2}} 1 F_1 (\frac{1}{2} + a, \frac{3}{2}, X) + \left( 1 + 2a \right) \frac{3}{3} X' \sqrt{\frac{nM_c}{2m_5} - \frac{im_5}{2} \tau} \right] \]

\[ \hat{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{u}_{-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ u_1 = \begin{pmatrix} 0 \\ 0 \\ -k_3 \end{pmatrix}, \quad u_{-1} = \begin{pmatrix} 0 \\ 0 \\ -k_1 + ik_2 \end{pmatrix} \]

where \( u_s \) and \( \hat{u}_s \) are column matrices (with spin quantum number \( s = \pm 1 \)) given as

\[ X = \pm \left( \frac{nM_c}{2m_5} - 2im_5 \right) \]

\[ S_1 = | (\pm \frac{nM_c}{4m_5} + im_5 - inM_c)_1 F_1 (a, \frac{1}{2}, X) \pm 2a \frac{nM_c}{2m_5} - im_5 )_1 F_1 (1 + a, \frac{3}{2}, X) | \]

Normalization constants \( N_1, N_2, N_3 \) and \( N_4 \) are determined through (3.12) as

\[ N_1 = \sqrt{M_c} \exp(\mp nM_c/2m_5) \]

\[ N_2 = \frac{\sqrt{M_c} \exp(\mp nM_c/2m_5)}{2\sqrt{\pi} | 1 F_1 (a, \frac{1}{2}, X) |} \]

\[ N_3 = \frac{\sqrt{M_c} \exp(\mp nM_c/2m_5)}{2\sqrt{\pi} s_1} \]

\[ N_4 = \frac{\sqrt{M_c} \exp(\mp nM_c/2m_5)}{2k\sqrt{\pi} s_2} \]

where

\[ \overline{X} = \pm \left( \frac{nM_c}{2m_5} - 2im_5 \right) \]
and
\[
S_2 = \left| \frac{(nM_c - 2im_5)}{\sqrt{2m_5}} \left( \pm \frac{nM_c}{4m_5} + \frac{im_5 + im_5 - inM_c}{1} \right) \right|
\]
\[
i \sqrt{\frac{m_5}{2}} F_1(\frac{1}{2} + a, \frac{3}{2}, X) + \frac{(1 + 2a)(nM_c - 2im_5)}{3} \sqrt{\frac{2m_5}{3}} F_1(\frac{3}{2} + a, \frac{5}{2}, X)
\]

Case II. When \( m_5 \gtrsim \frac{n^2}{2} M_e \), a new variable \( \tau' \) is defined as \( \tau' = m_5 - \frac{nM_c}{2} \). Now (3.14) is written as
\[
\frac{d^2 f_I}{d \tau^2} + \frac{4}{n^2 M_c^2} [k^2 + \frac{1}{n^2 M_c^2} (m_5 - \tau')^2 \tau^2 - \frac{i}{2} \frac{d}{d \tau} \{ \tau' (m_5 - \tau') \}] f_I = 0
\]
Since \( m_5 \gg \tau' \), so one gets
\[
\frac{d^2 f_I}{d \tau^2} + \left[ \frac{4}{n^2 M_c^2} (k^2 - \frac{im_5}{2}) + \frac{4m_5^2}{n^4 M_c^4} \tau^2 \right] f_I = 0
\]
having the exact solution
\[
f_I = \exp\left(-\frac{z}{2}\right) [\tilde{N}_1 \ F_1(k, \frac{1}{2}, z) + \tilde{N}_2 z^{1/2} \ F_1(k + 1, 3, z)]
\]
where
\[
z = \frac{im_5}{2} (1 - \frac{2m_5}{nM_c})^2
\]
and
\[
4k = 1 + \frac{2i}{m_5} (k^2 - \frac{im_5}{2})
\]
Connecting (3.6) and (3.22)
\[
f_{II} = \frac{i(k, \sigma)}{k^2} \exp\left(-\frac{z}{2}\right) [\tilde{N}_3 \left( \frac{1}{2} \frac{z'}{2} - \frac{im_5 \tau^2}{4} + \frac{im_5 \tau}{2} \right) F_1(k, \frac{1}{2}, z) +
2kz' \ F_1(k + 1, \frac{3}{2}, z)] + \tilde{N}_4 \left( \frac{1}{2} \frac{z'}{2} - \frac{im_5 \tau^2}{4} + \frac{im_5 \tau}{2} + \frac{z'}{2z} \right) F_1(k + 1, \frac{3}{2}, z) +
\]
\[
\frac{(2k + 1)}{3} \frac{z'}{2} z^{1/2} F_1(k + 3, \frac{5}{2}, z)]
\]
Corresponding to \( f_I \) and \( f_{II} \), \( \psi_{k\lambda I}^n \) and \( \psi_{k\lambda II}^n \) as are written as
\[
\psi_{k\lambda I}^n = (2\pi)^{-3/2} e^{i[k, k'] \left( \frac{\tau}{2} \right) + i(z - z')/2} \frac{i(k, \sigma)}{k^2} \times
\]
\[
\left[ \frac{1}{2} \frac{z'}{2} - \frac{im_5 \tau^2}{4} + \frac{im_5 \tau}{2} \right] F_1(k, \frac{1}{2}, z) + 2kz' \ F_1(k + 1, \frac{3}{2}, z)
\]
\[
+ \frac{(2k + 1)}{3} \frac{z'}{2} z^{1/2} F_1(k + 3, \frac{5}{2}, z)]
\]
(3.25)
On normalizing these solutions
\[ \tilde{N}_1 = \sqrt{M_c} [2k \sqrt{\pi} \ | \ _1 F_1(k, \frac{1}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) \ | ]^{-1} \]

\[ \tilde{N}_2 = n\sqrt{M_c} [2\sqrt{\pi}(nM_c - m_5)\sqrt{2m_5} \ | \ _1 F_1(k + \frac{1}{2}, \frac{3}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) \ | ]^{-1} \]

\[ \tilde{N}_3 = \frac{kn\sqrt{M_c}}{2\sqrt{n}}(\tilde{s}_1)^{-1} \]

\[ \tilde{N}_4 = \frac{kn\sqrt{M_c}}{2\sqrt{n}}(\tilde{s}_2)^{-1} \]

where
\[ \tilde{s}_1 = | \{ im_5(nM_c - m_5) - in^2M_c^2 + inm_5M_c \}_1 F_1(k, \frac{1}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2 \]

\[ + 4ikm_5(nM_c - m_5)_1 F_1(k + 1, \frac{3}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) | \]

and
\[ (\tilde{s}_2)^2 = m_5(1 - \frac{m_5}{nM_c}) | \{ \frac{im_5(nM_c - m_5)}{nM_c} - inM_c + im_5 + \frac{nM_c}{2(nM_c - m_5)} \} \times \]

\[ _1 F_1(k + \frac{1}{2}, \frac{3}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) + \]

\[ \frac{(2k + 1) 2im_5(nM_c - m_5)}{3nM_c} _1 F_1(k + \frac{3}{2}, \frac{5}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) | \]

\[ 3B, \ a(t) \approx t \]

From (3.3), in this case
\[ \tau = \ln t, \quad a(t) = \exp(\tau) \] (3.26)

Connecting (3.7) and (3.26)
\[ f''_I + [k^2 + e^{2\tau}(m_5 - nM_ce^{\tau})^2 + ie^{\tau}(m_5 - nM_e^{\tau}) - inM_e^{e\tau}]f_I = 0 \] (3.27)

**Case 1** When \( m_5 >> nM_e e^{\tau} \), (3.27) is approximated as
\[ f''_I + [k^2 + im_5 e^{\tau} + (m_5^2 - inM_e)e^{2\tau}]f_I = 0 \] (3.28)

which yields the exact solutions
\[ f_I(\tau) = \exp[\mp ik\tau + (inM_e - m_5^2)e^{\tau}] \times [c_1 \ _1 F_1(\frac{2l' + im_5}{2l'}, 2l, -e^{\tau}) + \]

\[ c_2 (-\frac{e^{\tau}}{2l'})^{1-2l} \ _1 F_1(\frac{2l' - 2l + im_5}{2l'}, 2 - 2l, -e^{\tau})] \] (3.29)
where
\[ l = \frac{1}{2} [1 \pm i2k], \quad l' = iM_e - m_\gamma^2 \]

Connecting (3.6) and (3.29)
\[ f_{II}(\tau) = \frac{i(k, \sigma)}{k^2} \exp[\mp ik\tau + (iM_e - m_\gamma^2)\tau^\tau] [c_3 X(\tau) + c_4 Y(\tau)] \quad (3.30) \]

where
\[
X(\tau) = \{ \mp ik + iM_e - m_\gamma^2 - i\tau (nM_e e^\tau - m_\gamma) \} \quad _{1}F_{1}(\frac{2l' + im_5}{2l'}, 2l, -\frac{e^\tau}{2l'})
- \frac{(2l' + im_5)}{8l'^2} \quad _{1}F_{1}(\frac{2l' + im_5 + 2l'}{2l'}, 1 + 2l, -\frac{e^\tau}{2l'})
\]

and
\[
Y(\tau) = (-\frac{e^\tau}{2l'})^{1-2l} \{1 - 2l \mp ik + iM_e - m_\gamma^2 - i\tau (nM_e e^\tau - m_\gamma) \} \times
_{1}F_{1}(\frac{2l' - 2l + 2l + im_5}{2l'}, 2 - 2l, -\frac{e^\tau}{2l'})
- \frac{(2l' - 2l + im_5)}{8l'^2(1 - l)} \quad _{1}F_{1}(\frac{4l' - 2l' + im_5}{2l'}, 3 - 2l, -\frac{e^\tau}{2l'})
\]

So,
\[
\psi_{k4I}s = (2\pi)^{-3/2} \exp[ik.x - \frac{9\tau}{4} \mp ik\tau + (iM_e - m_\gamma^2)\tau^\tau] \times
[c_1u_s \quad _{1}F_{1}(\frac{2l' + im_5}{2l'}, 2l, -\frac{e^\tau}{2l'})
+ c_2\hat{u}_s(-\frac{e^\tau}{2l'})^{1-2l} \quad _{1}F_{1}(\frac{2l' - 2l' + im_5}{2l'}, 2 - 2l, -\frac{e^\tau}{2l'})] \quad (3.31)
\]

and
\[
\psi_{k4II}s = \frac{i(k, \sigma)}{k^2} (2\pi)^{-3/2} \exp[ik.x - \frac{9\tau}{4} \mp ik\tau + (iM_e - m_\gamma^2)\tau^\tau] \times
[c_3u_s X(\tau) + c_4\hat{u}_s Y(\tau)] \quad (3.32)
\]

On normalising these solutions
\[
c_1 = \frac{\sqrt{M_e \exp(m_\gamma^2)}}{2\sqrt{i}k \quad | \quad _{1}F_{1}(\frac{2l' + im_5}{2l'}, 2l, -\frac{1}{2l'})|}
\]
\[
c_2 = \frac{\sqrt{M_e (2l')^{1-2l} \exp(m_\gamma^2)}}{2\sqrt{i} \quad | \quad _{1}F_{1}(\frac{2l' - 2l' + im_5}{2l'}, 2 - 2l, -\frac{1}{2l'})|}
\]
\[
c_3 = \frac{\sqrt{M_e \exp(m_\gamma^2)}}{2\sqrt{i} \quad | \quad X(0)|}
\]
\[ c_4 = \frac{\sqrt{M_e \exp(m^2_{\psi})}}{2\sqrt{\pi}} | Y(0) | \]

where \( X(\tau) \) and \( Y(\tau) \) are defined as in (3.30)

**Case II.** When \( m_5 \geq n M_c e^\tau \), (3.27) is approximated as

\[ f''_I + [k^2 - in M_c e^{2\tau}] f_I = 0 \quad (3.33) \]

which integrates to

\[
f_I = \exp(\pm i k \tau + L' e^\tau) [\tilde{c}_1 \ _1 F_1 \left( \frac{2LL' - in M_c}{2L'}, 2L, -2L' e^\tau \right) + \tilde{c}_2 (-2LL')^{1/2} e^{\tau(1-2l)} \ _1 F_1 \left( \frac{2LL' - in M_c}{2L'}, 2 - 2L, -2L' e^\tau \right)] \quad (3.34)
\]

where \( L = \frac{1}{2}[1 \pm 2k] \) and \( L' = \sqrt{n M_c} \times \exp[(2r + 1) \frac{\pi}{4}] \) with \( r = 0, 1, 2, \ldots \)

(3.6) and (3.34) yield

\[
f_{II} = \frac{i(k, \sigma)}{k^2} \exp(\pm ik \tau + L' e^\tau) [\tilde{c}_3 \tilde{X}(\tau) \quad \tilde{c}_4 \tilde{Y}(\tau)] \quad (3.35)
\]

where

\[
\tilde{X}(\tau) = \{ \pm ik \tau + L' e^\tau + i e^\tau (m_5 - n M_c e^\tau) \}_{1} F_1 \left( \frac{2LL' - in M_c}{2L'}, 2L, -2L' e^\tau \right) - \frac{(2LL' - in M_c)}{2L} e^\tau \ _1 F_1 \left( \frac{2LL' + 2LL' - in M_c}{2L'}, 1 + 2L, -2L' e^\tau \right)
\]

and

\[
\tilde{Y}(\tau) = \{ \pm ik \tau + L' e^\tau + (-2LL')^{1/2} (2 - 2L) e^{\tau(1-2l)} + i e^\tau (m_5 - n M_c e^\tau) \} \times
\ _1 F_1 \left( \frac{2LL' - 2LL' - in M_c}{2L'}, 2 - 2L, -2L' e^\tau \right) + (-2LL')^{1/2} (2LL' - in M_c) e^{2(1-L)\tau} \times \\
\ _1 F_1 \left( \frac{4LL' - 2LL' - in M_c}{2L'}, 3 - 2L, -2L' e^\tau \right)
\]

So,

\[
\psi_{k4ls}^n = (2\pi)^{-3/2} \exp[\pm \frac{9\pi}{4} i k \tau + L' e^\tau] \times \\
[\tilde{c}_1 u_s \ _1 F_1 \left( \frac{2LL' - in M_5}{2L'}, 2L, -2L' e^\tau \right) + \tilde{c}_2 \ _1 F_1 \left( \frac{2LL' - 2LL' - in M_c}{2L'}, 2 - 2L, -2L' e^\tau \right)] \quad (3.36)
\]
and

\[
\psi_{k4\ell}^n = \frac{i(k,\sigma)}{k^2} (2\pi)^{-3/2} \exp[ik.x - \frac{9\tau}{4} \pm ik\tau + L'e^\tau] \times \\
\left[ \tilde{c}_3 u_s \tilde{X}(\tau) + \tilde{c}_4 \hat{u}_s \tilde{Y}(\tau) \right]
\]

(3.37)

where \(\tilde{X}(\tau)\) and \(\tilde{Y}(\tau)\) are defined in (3.35)

On normalizing the above solutions

\[
\begin{align*}
\tilde{c}_1 &= \frac{e^{-L'}\sqrt{M_c}}{k\sqrt{2\pi}} | \text{I}_F(2L' - \text{inM}_c, 2L', -2L') | \\
\tilde{c}_2 &= \frac{e^{-L'}\sqrt{M_c}(-2L')^{L-1}}{\sqrt{2\pi}} | \text{I}_F(2L' - 2L' - \text{inM}_c, 2 - 2L, -2L') | \\
\tilde{c}_3 &= \frac{e^{-L'}\sqrt{M_c}}{\sqrt{2\pi}} | X(0) | \\
\tilde{c}_4 &= \frac{ke^{-L'}\sqrt{M_c}}{\sqrt{2\pi}} | Y(0) |
\end{align*}
\]

From (3.3)

\[
-x\tau = e^{-x\tau}, \quad a(\tau) = -(x\tau)^{-1}
\]

(3.38)

so, (3.7) reduces to

\[
f''_I + \left[ k^2 + \frac{1}{x^2\tau^2}(m_5 + \frac{nM_c}{x\tau})^2 - i \frac{d}{d\tau} \left\{ \frac{1}{x\tau}(m_5 + \frac{nM_c}{x\tau}) \right\} \right] f_I = 0
\]

(3.39)

**Case 1** When \(m_5 >> \frac{nM_c}{x\tau}\) (3.39) is approximated to

\[
\tau^2 f''_I + \left[ k^2 \tau^2 + \frac{m_5(m_5 + ix)}{x^2} \right] f_I = 0
\]

(3.40)

having exact solution

\[
f_I = \exp[\pm ik\tau + \frac{1}{2}(1 \pm i\sqrt{4a^2 - 1})ln\tau] [D_1 1F_1(\frac{a}{2}, 2g, -2g'\tau) \\
+ D_2(-2g')^{\pm i\sqrt{4a^2 - 1}} 1F_1(1 + \frac{a}{2} - 2g, 2 - 2g, -2g')]
\]

(3.41)

where

\[
a = m_5(m_5 + ix)x^{-2}, \quad g = \frac{1}{2}[1 \pm \sqrt{4a^2 - 1}]
\]
and \( g' = \mp ik \).

Connecting (3.6) and (3.41)

\[
f_{II} = \frac{i(k, \sigma)}{k^2} \exp[\pm i k \tau + \frac{1}{2}(1 \pm i \sqrt{4a^2 - 1})\ln \tau] [D_3 X_1(\tau) + D_4 Y_1(\tau)]
\]

where

\[
X_1(\tau) = \left\{ \frac{1 \pm \sqrt{4a^2 - 1}}{2\tau} \right\} \pm ik - \frac{i}{x\tau} \left( \frac{n M_c}{x \tau} + m_5 \right) - \frac{ag'}{2g} \left\{ \frac{1}{2}, 2g - 2g' \tau \right\}
\]

and

\[
Y_1(\tau) = (-2g' \tau)^{\pm i \sqrt{4a^2 - 1}} \left\{ \frac{1 \pm i \sqrt{4a^2 - 1}}{2\tau} \right\} \pm ik - \frac{i}{x\tau} \left( \frac{n M_c}{x \tau} + m_5 + \tau^{-1} \right) \times
\]

\[
1 F_1(1 + a, -2g, 2, -2g, -2g' \tau) - \frac{(2 + a - 4g)}{2(1 - g)} g' \left\{ \frac{1}{2}, 2g - 2g' \tau \right\}
\]

So,

\[
\psi_{kII}^n = (2\pi)^{-3/2} e^{i(k, x)} (-\frac{1}{x\tau})^{9/4} \exp[\pm ik \tau + \frac{1}{2}(1 \pm i \sqrt{4a^2 - 1})\ln \tau] \times
\]

\[
[D_1 u_s \left\{ \frac{1}{2}, 2g, -2g' \tau \right\} + 
D_2 \hat{u}_s (-2g' \tau)^{\pm i \sqrt{4a^2 - 1}} 1 F_1(\left\{ \frac{2 + a - 4g}{2}, 2 - 2g, -2g' \tau \right\}]
\]

and

\[
\psi_{kIII}^n = (2\pi)^{-3/2} e^{i(k, x)} (-\frac{1}{x\tau})^{9/4} \exp[\pm ik \tau + \frac{1}{2}(1 \pm i \sqrt{4a^2 - 1})\ln \tau] \times
\]

\[
\frac{i(k, \sigma)}{k^2} \left\{ D_3 u_s X_1(\tau) + D_4 \hat{u}_s Y_1(\tau) \right\}
\]

Normalization of these solutions yields

\[
D_1 = \sqrt{M_c x} e^{-i\pi/2}[2\sqrt{\pi} k \left\{ 1 F_1(\left\{ \frac{1}{2}, 2g, \frac{2g'}{x} \right\} \right\}^{-1}
\]

\[
D_2 = \sqrt{M_c x} e^{-i\pi/2}[2\sqrt{\pi} \left\{ 1 F_1(\left\{ \frac{2 + a - 4g}{2}, 2 - 2g, \frac{2g'}{x} \right\} \right\}^{-1}
\]

\[
D_3 = \sqrt{M_c x} e^{-i\pi/2}[2\sqrt{\pi} \left\{ X_1(\left\{ \frac{1}{x} \right\} \right\}^{-1}
\]

and

\[
D_4 = k \sqrt{M_c x} e^{-i\pi/2}[2\sqrt{\pi} \left\{ Y_1(\left\{ \frac{1}{x} \right\} \right\}^{-1}
\]

Case II When \( m_5 \geq n M_c x \tau \), (3.39) approximates to

\[
\tau^2 f''_I + [\alpha + \beta \tau + \gamma \tau^2] f_I = 0
\]
where \( \alpha = m_5^2 e^{-2} \), \( \beta = m_3^2 (2 - i x)(nM_c)^{-1} \)
and \( \gamma = k^2 + m_3^2 (m_5 - i x)(nM_c)^{-2} \)
(3.45) yields the solution

\[
f_f = \exp[\pm i \sqrt{\gamma} \tau + \frac{1}{2} (1 \pm i \sqrt{4 \alpha - 1}) \ln \tau] \times [\tilde{D}_1 \, _1F_1(\frac{2jj' + \beta}{2j'}, 2j, -2j' \tau) + \tilde{D}_2 (-2j' \tau)^{1-2j} \, _1F_1(\frac{2jj' - 2jj' + \beta}{2j'}, 2 - 2j, -2j' \tau)]
\]

(3.46)

where \( j = \frac{1}{2} [1 \pm i \sqrt{4 \alpha - 1}] \) and \( j' = \pm i \sqrt{\gamma} \)
Connecting (3.6) and (3.46)

\[
f_{II} = \frac{i (k, \sigma)}{k^2} \exp[\pm i \sqrt{\gamma} \tau + \frac{1}{2} (1 \pm i \sqrt{4 \alpha - 1}) \ln \tau] \times [\tilde{D}_3 X_2(\tau) + \tilde{D}_4 Y_2(\tau)]
\]

(3.47)

where

\[
X_2(\tau) = \{ \frac{1}{2\tau} (1 \mp i \sqrt{4 \alpha - 1}) \mp i \sqrt{\gamma} - \frac{i}{\tau} (nM_c / xT + m_5) \}
\]

\[
_1F_1(\frac{2jj' + \beta}{2j'}, 2j, -2j' \tau) - \frac{(2jj' + \beta)}{2j} \, _1F_1(\frac{2jj' + \beta + 2j'}{2j'}, 1 + 2j, -2j' \tau)
\]

and

\[
Y_2(\tau) = \{ \frac{1}{2\tau} (1 \mp i \sqrt{4 \alpha - 1}) \mp i \sqrt{\gamma} - \frac{i}{\tau} (nM_c / xT + m_5) \} (-2j' \tau)^{1-2j} \times
\]

\[
_1F_1(\frac{2jj' - 2jj' + \beta}{2j'}, 2 - 2j, -2j' \tau) - 2j(1 - 2j)(-2j' \tau)^{-2j} \times
\]

\[
_1F_1(\frac{2jj' - 2jj' + \beta}{2j'}, 2 - 2j, -2j' \tau)
\]

\[
- (-2j' \tau)^{1-2j} \, \frac{(2jj' - 2jj' + \beta)}{2(1 - j)} \, _1F_1(\frac{4jj' - 2jj' + \beta}{2j'}, 3 - 2j, -2j' \tau)
\]

So

\[
\psi_{kIA}^I = (2\pi)^{-3/2} e^{i(k, x)} \exp[\pm i \sqrt{\gamma} \tau + \frac{1}{2} (1 \pm i \sqrt{4 \alpha - 1}) \ln \tau] \times
\]

\[
[\tilde{D}_1 \, u_s \, _1F_1(\frac{2jj' + \beta}{2j'}, 2j, -2j' \tau) + \tilde{D}_2 \, \hat{u}_s (-2j' \tau)^{1-2j} \, _1F_1(\frac{2jj' - 2jj' + \beta}{2j'}, 2 - 2j, -2j' \tau)]
\]

(3.48)

and

\[
\psi_{kIA}^I = (2\pi)^{-3/2} e^{ikx} \frac{i (k, \sigma)}{k^2} \exp[\pm i \sqrt{\gamma} \tau + \frac{1}{2} (1 \pm i \sqrt{4 \alpha - 1}) \ln \tau] \times
\]

\[
[\tilde{D}_3 \, u_s X_2(\tau) + \tilde{D}_4 \, u_s Y_2(\tau)]
\]

(3.49)
On normalization of these solutions

\[ \hat{D}_1 = \sqrt{M_c} e^{-\nu x/2} \left[ 2k \sqrt{\pi} \right] \frac{1}{\Gamma(\frac{2j+\beta}{2})} \frac{2j-2j^2 + \beta}{4j^2} \left[ 2j - 2j^2 + \beta \right] \left( \frac{2j}{x} \right) \left[ 2k \sqrt{\pi} \right]^{-1} \]

\[ \hat{D}_2 = \sqrt{M_c} e^{-\nu x/2} \left[ 2\sqrt{\pi} \right] \frac{1}{\Gamma(\frac{2j+\beta}{2})} \frac{2j-2j^2 + \beta}{4j^2} \left[ 2j - 2j^2 + \beta \right] \left( \frac{2j}{x} \right) \left[ 2\sqrt{\pi} \right]^{-1} \]

\[ \hat{D}_3 = \sqrt{M_c} e^{-\nu x/2} \left[ 2\sqrt{\pi} \right] \frac{1}{\Gamma(\frac{2j+\beta}{2})} \frac{2j-2j^2 + \beta}{4j^2} \left[ 2j - 2j^2 + \beta \right] \left( \frac{2j}{x} \right) \left[ 2\sqrt{\pi} \right]^{-1} \]

and

\[ \hat{D}_4 = \sqrt{M_c} e^{-\nu x/2} \left[ 2\sqrt{\pi} \right] \frac{1}{\Gamma(\frac{2j+\beta}{2})} \frac{2j-2j^2 + \beta}{4j^2} \left[ 2j - 2j^2 + \beta \right] \left( \frac{2j}{x} \right) \left[ 2\sqrt{\pi} \right]^{-1} \]

4. Current for \( \psi_4^n \)

The current is defined as

\[ J_4^{\mu n} = \bar{\psi}_4^n \gamma^\mu \psi_4^n, \quad (\mu = 0, 1, 2, 3) \quad (4.1) \]

which is divergence - free as \( J_4^{\mu n}; \mu = 0 \). For a massive field

\[ J_4^{\mu n} = \bar{\psi}_4^n \gamma^\mu \psi_4^n = \frac{1}{2M} \bar{\psi}_4^n (i\partial_\lambda \gamma^\lambda \gamma^\mu - i\gamma^\lambda \gamma^\mu \partial_\lambda - i[\gamma^\lambda \Gamma_\lambda, \gamma^\mu]) \psi_4^n \quad (4.2) \]

where \( M = m_5 - a(t)nM_c \)

which can be re-expressed as

\[ J_4^{\mu \bar{n}} = \frac{1}{2M} \bar{\psi}_4^n (i\partial_\lambda \gamma^\lambda \gamma^\mu - i\gamma^\lambda \gamma^\mu \partial_\lambda - i[\gamma^\lambda \Gamma_\lambda, \gamma^\mu]) \psi_4^n \]

where

\[ \bar{\psi}_4^n \partial_\lambda \psi_4^n = \bar{\psi}_4^n \partial_\lambda \psi_4^n - \psi_4^n \partial_\lambda \bar{\psi}_4^n, \quad M = m_5 - \frac{a(t)n}{k_c} \]

(here \( \mu, \nu, \lambda, \ldots \) run from 0 to 3)

In the \( M^4 \) spacetime

\[ \gamma^{\lambda, \lambda} = 0, \quad [\gamma^{\lambda, \gamma^i}, \gamma^{\lambda, \lambda}] = [\gamma^0, \gamma^i](-\bar{a}a^2) \quad (i = 1, 2, 3) \]

\[ [\gamma^0, \gamma^0, 0] = 0, \quad \sigma^{0i} = \frac{i}{2a} [\gamma^0, \gamma^i], \quad \sigma^{ij} = \frac{i}{2a^2} [\gamma^i, \gamma^j] \]

\[ \Gamma_0 = 0, \quad \Gamma_1 = \dot{a} \gamma^1 \gamma^0, \quad \Gamma_2 = \dot{a} \gamma^2 \gamma^0 \quad \text{and} \quad \Gamma_3 = \dot{a} \gamma^3 \gamma^0 \]
So

\[ J^{\text{no}}_4 = \frac{1}{2M} \left( \bar{\psi}^n_4 \sigma^{\text{no}} \psi^n_4 \right)_\alpha - \frac{i}{4M} \bar{\psi}^n_4 \frac{\partial_0}{\partial t} \psi^n_4 \]

and

\[ J^{ni}_4 = \frac{1}{2M} \partial_0 (\bar{\psi}^n_4 \sigma^i \psi^n_4) + \frac{1}{2M} \partial_j (\bar{\psi}^n_4 \sigma^{ji} \psi^n_4) + \frac{7i}{2M} \left( \frac{\dot{a}}{a^2} \right) \bar{\psi}^n_4 \gamma^0 \gamma^i \psi^n_4 + \frac{i}{4Ma^2} \bar{\psi}^n_4 \partial_i \psi^n_4 \]

In terms of polarization density and magnetization density \(j^{\text{no}}\) and \(j^{ni}\) is written as

\[ j^{\text{no}}_4 = \vec{\nabla} \cdot \vec{p}_4^n + \rho_{4(\text{convective})}^n \]

and

\[ j^{ni}_4 = \partial_t \vec{p}_4^n + \vec{\nabla} \times \vec{M}_4^n + \gamma^n_{4(\text{convective})} + 7 \left( \frac{\dot{a}}{a} \right) \vec{p}_4^n \]

When \(m_5 >> \frac{an}{k_c}\), \(M \simeq m_5\), so \(P_4^{\text{in}} = \frac{i}{2m_5a} \bar{\psi}^n_4 \gamma^i \gamma^\alpha \psi^n_4\)

and

\[ M_4^{\text{in}} = \epsilon^{ijk} \left( \frac{i}{4m_5a^2} \right) \bar{\psi}^n_4 [\gamma_j, \gamma_k] \psi^n_4 \]

\[ \rho_{4(\text{convective})}^n = - \frac{i}{4m_5} \bar{\psi}^n_4 \frac{\partial_0}{\partial t} \psi^n_4 \]

and

\[ j^{\text{in}(\text{convective})}_4 = - \frac{i}{4m_5a^4} \bar{\psi}^n_4 \partial_i \psi^n_4 \]

But when \(m_5 \simeq \frac{an}{k_c}\), one has

\[ P_4^{\text{in}} = \frac{i}{2(m_5 - \frac{an}{k_c})a} \bar{\psi}^n_4 \gamma^i \gamma^\alpha \psi^n_4 \]

\[ M_4^{\text{in}} = \epsilon^{ijk} \left( \frac{i}{4(m_5 - \frac{an}{k_c})a^2} \right) \bar{\psi}^n_4 [\gamma_j, \gamma_k] \psi^n_4 \]

\[ \rho_{4(\text{convective})}^n = - \left[ \frac{i}{4(m_5 - \frac{an}{k_c})a^4} \right] \bar{\psi}^n_4 \partial_0 \psi^n_4 \]

and

\[ j^{\text{in}(\text{convective})}_4 = \frac{i an}{2a(m_5R - an)} \bar{\psi}^n_4 \gamma^i \gamma^\alpha \psi^n_4 - \frac{iR}{4(m_5R - an)a^4} \bar{\psi}^n_4 \partial_i \psi^n_4 \]

Thus, it is found that polarization vector, magnetization density (which is a pseudo-vector), \(\rho_{(\text{convective})}\) and \(j_{(\text{convective})}\) depend on time. It is interesting to note that when \(m_5 \simeq \frac{an}{k_c}\) (which yields realistic fermions) \(j_{(\text{convective})}\) contains an extra term \(\frac{i an}{2a(m_5R - an)} \bar{\psi}^n_4 \gamma^i \gamma^\alpha \psi^n_4\).
References


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