SOME COMMUTATIVITY RESULTS FOR ONE SIDED \( s \)-UNITAL RINGS

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Summary: Let \( R \) be an associative ring, \( Z[t] \) is the totality of polynomials in \( t \) with coefficients in \( Z \), the ring of integers, and let \( A \) be any non-empty subset of \( R \). In this paper, we consider the following ring properties:

\((H)\): For each \( x, y \) in \( R \), there exists \( f(t) \in t^1 Z[t] \) such that \([x−f(x), y] = 0\).

\((C)\): For all \( x, y \) in \( R \), there exist \( f(t), g(t) \in t^1 Z[t] \) such that \([x−f(x), y−g(y)] = 0\).

\((I−A)\): For each \( x \) in \( R \) either \( x \) is central or there exists \( f(t) \in t^1 Z[t] \) such that \( x−f(x) \in A \).

\(P\left(m, n, p, q\right)\): For each \( x, y \) in \( R \), there exists \( f(t) \in t^1 Z[t] \) such that \([x^m yx^n − x^p f(y) x^q, x] = 0\), where \( m, n, p, q \) are fixed non-negative integers.

\(P^*\left(m, n, p, q\right)\): For each \( x, y \) in \( R \), there exist integers \( m\geq0, n\geq0, p\geq0, q\geq0 \) and \( f(t) \in t^2 Z[t] \) such that \([x^m yx^n − x^p f(y) x^q, x] = 0\).

In fact, we prove "If \( R \) is a left (resp. right) \( s \)-unital ring satisfying \( P\left(m, 0, p, q\right) \) (resp. \( P^*\left(0, n, p, q\right) \)), then \( R \) is commutative (and conversely)",

and "If \( R \) is a left (resp. right) \( s \)-unital ring satisfying \( P^*\left(m, 0, p, q\right) \) (resp. \( P^*\left(0, n, p, q\right) \)) and \( (I−N(R)) \), then \( R \) is commutative (and conversely)".

TEK YANLI "s-UNITAL" HALKALAR IÇİN BAZI KOMUTATİFLİK SONUÇLARI

Özet: \( R \) asosyatif bir halka, \( Z[t] \) \( t \) nin katsayıları \( Z \) tamsayalar halkasından alınmış bütün polinomlarından oluşan halka, \( A \) da \( R \) nin boş olmayan herhangi bir alt kümesi olsun. Bu çalışmada

\((H)\): Her \( x, y \in R \) için, \([x−f(x), y] = 0\) olacak şekilde \( f(t) \in t^1 Z[t] \) vardır.

\((C)\): Bütün \( x, y \in R \) için, \([x−f(x), y−g(y)] = 0\) olacak şekilde \( f(t), g(t) \in t^1 Z[t] \) vardır.

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(I—A) : Her \( x \in R \) için \( x \) merkeze aitir veya \( x - f(x) \in A \) olacak şekilde \( f(t) \in \mathbb{Z}[t] \) vardır.

\[ P(m, n, p, q) : \text{Her } x, y \in R \text{ için, } x^m y^n - x^p f(y) x^q, x = 0 \text{ olacak şekilde } f(t) \in t^q \mathbb{Z}[t] \text{ vardır (burada } m, n, p, q \text{ negatif olmayan sabit tam sayılardır).} \]

\[ P^*(m, n, p, q) : \text{Her } x, y \in R \text{ için, } x^m y^n - x^p f(y) x^q, x = 0 \text{ olacak şekilde } m \geq 0, n \geq 0, p \geq 0, q \geq 0 \text{ tam sayıları ve } f(t) \in t^q \mathbb{Z}[t] \text{ vardır.} \]

Gibi halka özellikleri kullanılarak şunlar ispat edilmektedir:

1) "R, \( P(m, 0, p, q) \) (\( P^*(0, n, p, q) \)) özelliğini gerçekleyen bir sol (sağ) \( s \)-unital halka ise \( R \) komütatif ve bunun karşısı da doğrudur".

2) "R, \( P^*(m, 0, p, q) \) özelliğini (\( P^*(0, n, p, q) \) ve \( I - N(R) \) özelliklerini) gerçekleyen bir sol (sağ) \( s \)-unital halka ise \( R \) komütatif ve bunun karşısı da doğrudur".

Let \( R \) be an associative ring (not necessarily with unity 1). A ring \( R \) is called left (resp. right) \( s \)-unital if \( x \in Rx \) (resp. \( x \in xR \)) for every \( x \in R \). Further, \( R \) is called \( s \)-unital if \( x \in Rx \cap xR \) for all \( x \in R \). If \( R \) is \( s \)-unital (resp. left or right \( s \)-unital), then for any finite subset \( F \) of \( R \), there exists an element \( e \in R \) such that \( ex = xe = x \) (resp. \( ex = x \) or \( xe = x \)) for all \( x \in F \). Such an element \( e \) will be called a pseudo (resp. pseudo left or pseudo right) identity of \( F \) in \( R \).

Throughout the paper \( Z(R) \) will denote the center of \( R \), \( N(R) \) the set of nilpotent elements of \( R \), \( C(R) \) the commutator ideal of \( R \), and \( A \) a non-empty subset of \( R \). As usual \( Z[t] \) is the totality of polynomials in \( t \) with coefficients in \( Z \), the ring of integers, and for any \( x, y \in R \), \( [x, y] = xy - yx \).

By \( GF(q) \), we mean the Galois field (finite field) with \( q \) elements, and \( (GF(q))_2 \) the ring of all \( 2 \times 2 \) matrices over \( GF(q) \). Set \( e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \text{ and } e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \) in \( (GF(p))_2 \) for a prime \( p \).

Now, we consider the following types of rings:

(i) \( \begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix} \), \( p \) a prime.

(ii) \( \begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix} \), \( p \) a prime.

(iii) \( \begin{pmatrix} 0 & GF(p) \\ GF(p) & 0 \end{pmatrix} \), \( p \) a prime.

(ii) \( M_n(K) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \right\} \), where \( K \) is a finite field with a non-trivial automorphism \( \sigma \).
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(iii) A non-commutative division ring.
(iv) \( S = < 1 > + T \), where \( T \) is a non-commutative radical subring of \( S \).
(v) \( S = < 1 > + T \), where \( T \) is a non-commutative subring of \( S \) such that \( T[T, T] = [T, T] T = 0 \).

From the proof of [19, Korollar 1] it can be easily seen that if \( R \) is a non-commutative ring with unity 1, then there exists a factorsubring of \( R \), which is of type (i), (ii), (i), (iii), (iv) or (v). This gives the following Meta Theorem, which plays the key role in our subsequent study:

Meta Theorem. Let \( P \) be a ring property which is inherited by factorsubrings. If no rings of type (i), (ii), (i), (iii), (iv) or (v) satisfy \( P \), then every ring with unity 1 satisfying \( P \) is commutative.

Next, we consider the following ring properties:

\( (H) \) : For each \( x, y \) in \( R \), there exists \( f(t) \in \mathbb{Z}[t] \) such that \( [x - f(x), y] = 0 \).

\( (C) \) : For all \( x, y \) in \( R \), there exist \( f(t), g(t) \in \mathbb{Z}[t] \) such that \( [x - f(x), y - g(y)] = 0 \).

\( (I - A) \) : For each \( x \) in \( R \) either \( x \) is central or there exists \( f(t) \in \mathbb{Z}[t] \) such that \( x - f(x) \in A \).

\( P(m, n, p, q) \) : For each \( x, y \) in \( R \), there exists \( f(t) \in \mathbb{Z}[t] \) such that \( x^m y x^n - x^p f(y) x^q, x = 0 \), where \( m, n, p, q \) are fixed non-negative integers.

\( P^*(m, n, p, q) \) : For each \( x, y \) in \( R \), there exist integers \( m \geq 0, n \geq 0, p \geq 0, q \geq 0 \) and \( f(t) \in \mathbb{Z}[t] \) such that \( x^m y x^n - x^p f(y) x^q, x = 0 \).

A well-known theorem of Herstein [10] (signified as Theorem \( H \)) asserts that every ring satisfying \( (H) \) is commutative. Recently, various authors have studied commutativity of rings satisfying conditions \( (C) \), but always under some restrictions (cf. [9], [13] & [15] etc.). More recently, Komatsu et al. [13] investigated the commutativity of rings satisfying the condition \( P^*(m, 0, 0, q) \). Further, in a paper [16] Nishinaka established the commutativity of ring \( R \) with the conditions \( P(m, 0, 0, q) \) and \( P(m, 0, p, 0) \). In fact, he proved that a ring \( R \) with unity 1 satisfying any one of the conditions \( P(m, 0, 0, q) \) and \( P(m, 0, p, 0) \) must be commutative. In the present paper, first we shall study the commutativity of rings satisfying \( P(m, n, p, q) \) and establish the commutativity of one sided \( s \)-unital ring with either of the conditions \( P(m, 0, p, q) \) and \( P(0, n, p, q) \). We then proceed to investigate the commutativity of rings satisfying \( P^*(m, 0, p, q) \) or \( P^*(0, n, p, q) \) together with the condition \( (C) \). As corollaries to our theorems we shall give several results concerning the commutativity of ring \( R \). The results obtained in sequel generalize [1, Theorem 1.1], [2, Theorems 2&3], [3, Theorems 1&2],
[4, Theorem], [5, Theorem 2], [6, Theorems 1-4], [7, Theorems 4 & 5], [15 Theorems 2 & 3 (2)], [16, Theorem 1], [18, Theorem] and [20, Theorem 2 (5)], and thus provide an effective measure to determine the commutativity of $R$.

We begin with the following lemmas, which are essentially proved in [13] and [15] respectively.

**Lemma 1** [13, Corollary 1]. Let $R$ be a ring with unity 1 satisfying (C). If $R$ is non-commutative, then there exists a factorsubring of $R$, which is of type (i) or (ii).

**Lemma 2** [15, Lemma 1]. If $R$ is left $s$-unital and not right $s$-unital, then $R$ has a factorsubring of type (i).

We pause to remark that the dual of Lemma 2 asserts that if $R$ is right $s$-unital and not left $s$-unital, then $R$ has a factorsubring of type (i).

The following proposition is an important one from the point of view that it serves as the foundation for our entire discussion.

**Proposition 1.** Let $R$ be a ring with unity 1. If $R$ satisfies $P(m, n, p, q)$, then there exists no factorsubring of $R$ which is of type (ii), (iii), (iv) or (v).

**Proof.** Consider the ring $M_2(K)$, a ring of type (ii). Let $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$. Then

$[x^m y x^n - x^p f(y) x^q, x] = - x^m [x, y] x^n = - \alpha^m (\alpha - \sigma(\alpha))^n y \neq 0$

for every $f(t) \in t^2 \mathbb{Z}[t]$. Thus no rings of type (ii) satisfy $P(m, n, p, q)$.

Next, if $R$ is a ring of type (iii), then choose $f(t) \in t^2 \mathbb{Z}[t]$ such that

$[x^{-m} y x^{-n} - x^{-p} f(y) x^{-q}, x^{-1}] = 0$.

This yields that $[x^{-m} y x^{-n} - x^{-p} f(y) x^{-q}, x] = 0$, that is

$x^{-m} [x, y] x^{-n} = x^{-p} [x, f(y)] x^{-q}$.

It follows that

$x^p [x, y] x^q = x^m [x, f(y)] x^q$.  \hspace{1cm} (1)

Now, choose $g(t) \in t^2 \mathbb{Z}[t]$ such that $[x^m f(y) x^n - x^p g(f(y)) x^q, x] = 0$. Hence we get

$x^m [x, f(y)] x^q = x^p [x, g(f(y))] x^q$.  \hspace{1cm} (2)

Comparing of (1) and (2) yields that $x^p [x, y] x^q = x^p [x, h(y)] x^q$, where $h(t) = g(f(t)) \in t^2 \mathbb{Z}[t]$. But, since $x$ is unit, $[y - h(y), x] = 0$ and by Theorem H, $R$ is commutative, a contradiction. Hence no rings of type (iii) satisfy $P(m, n, p, q)$. 

Further, suppose that $R$ has a factorsubring of type (iv). Let $a, b \in T$. Since $1 - a$ is a unit, there exists $f(t) \in \mathbb{Z}[t]$ such that $[a, b - f(b)] = -[1 - a, b - f(b)] = 0$, by above paragraph. Hence, by Theorem H, $T$ is commutative. This is impossible. Hence no rings of type (iv) satisfy $P(m, n, p, q)$.

Finally, suppose that $R$ is of type (v). For each $a, b \in T$, there exists $f(t) \in \mathbb{Z}[t]$ such that

$$[a, b] = (a + 1)^m [a, b] (a + 1)^n = (a + 1)^p [a, f(b)] (a + 1)^q = 0.$$  

This is a contradiction.

Hence, it proves that no rings of type (ii), (iii), (iv) or (v) satisfy $P(m, n, p, q)$.

Lemma 3. Let $R$ be a ring with unity 1. If for each $x, y$ in $R$ there exists an integer $k=k(x, y) > 1$ such that $x^k [x, y] = 0$ or $[x, y] x^k = 0$, then necessarily $[x, y] = 0$.

Proof. Choose an integer $k_1 = k(x, y) > 1$ such that $(x+1)^{k_1} [x, y] = 0$. Now, if $N = \max(k, k_1)$, then it follows that $x^N [x, y] = 0$ and $(x+1)^N [x, y] = 0$. We have $[x, y] = [(x+1)-x]^{2N+1} [x, y]$. On expanding the expression on right hand side by binomial theorem and using the fact that $x^N [x, y] = 0$ and $(x+1)^N [x, y] = 0$, we get $[x, y] = 0$. Similarly, if $[x, y] x^k = 0$, then using the same techniques, we get the required result.

Lemma 4. Let $R$ be a ring with 1 satisfying any one of the properties $P^*(m, 0, p, q)$ and $P^*(0, n, p, q)$. Then $N(R) \subseteq Z(R)$.

Proof. Property $P^*(m, 0, p, q)$ may be written as $x^m [x, y] - x^p [x, f(y)] x^q = 0$. Let $a \in N(R)$ and $x$ be an arbitrary element of $R$. Then there exist integers $m_1 = m(x, a) \geq 0$, $p_1 = p(x, a) \geq 0$, $q_1 = q(x, a) \geq 0$ such that $x^{m_1} [x, a] = x^{p_1} [x, f_1(a)] x^{q_1}$ for some $f_1(t) \in \mathbb{Z}[t]$. Similarly, for the pair of elements $x, f_1(a)$, there exist integers $m_2 = m(x, f_1(a)) \geq 0$, $p_2 = p(x, f_1(a)) \geq 0$, $q_2 = q(x, f_1(a)) \geq 0$ such that

$$x^{m_2} [x, f_1(a)] = x^{p_2} [x, f_2(f_1(a))] x^{q_2},$$

for some $f_2(t) \in \mathbb{Z}[t]$, which yields that

$$x^{m_1+m_2} [x, a] = x^{p_1+p_2} [x, f_2(f_1(a))] x^{q_1+q_2}.$$  

Thus, it is clear that for an arbitrary $k$, there exist integers $m_1, m_2, \ldots, m_k \geq 0$, $p_1, p_2, \ldots, p_k \geq 0$, and $q_1, q_2, \ldots, q_k \geq 0$ such that

$$x^{m_1+m_2+\ldots+m_k} [x, a] = x^{p_1+p_2+\ldots+p_k} [x, f_k(\ldots(f_1(a) \ldots)) \ldots] x^{q_1+q_2+\ldots+q_k}.$$  

But since $a$ is nilpotent, $x^{m_1+m_2+\ldots+m_k} [x, a] = 0$ for sufficiently large $k$. Hence in view of Lemma 3, we get $[x, a] = 0$ for all $x$ in $R$. This proves that $N(R) \subseteq Z(R)$.

Using the similar arguments one can establish the result if $R$ satisfies $P^*(0, n, p, q)$. 

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Following [11], let $P$ be a ring property. If $P$ is inherited by every subring and every homomorphic image, then $P$ is called an $h$-property. More weakly, if $P$ is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then $P$ is called an $H$-property.

A ring property $P$ such that a ring $R$ has the property $P$ if and only if all its finitely generated subrings have $P$, is called an $F$-property.

**Proposition 2** [11, Proposition 1]. Let $P$ be an $H$-property, and let $P'$ be an $F$-property. If every ring $R$ with unity $1$ having the property $P$ has the property $P'$, then every $s$-unital ring having $P$ has $P'$.

We are now well-equipped to prove the following:

**Theorem 1.** If $R$ is a left $s$-unital ring satisfying $P(m, 0, p, q)$, then $R$ is commutative (and conversely).

**Proof.** Consider the ring of type (i). Then
\[
[(e_{11} + e_{12})^m e_{12} - (e_{11} + e_{12})^p f(e_{12}) (e_{11} + e_{12})^q, e_{11} + e_{12}] = -e_{12} \neq 0,
\]
for all integers $m \geq 0$, $p \geq 0$, $q \geq 0$ and $f(t) \in t^2 \mathbb{Z}[t]$. Accordingly, $R$ has no factorsubrings of type (i). Hence by Lemma 2, $R$ is $s$-unital and in view of Proposition 2, we may assume that $R$ has unity $1$.

Combining the above fact with Proposition 1, we see that no rings of type (i), (ii), (iii), (iv) or (v) satisfy the ring property $P(m, 0, p, q)$ and hence by Meta Theorem, $R$ is commutative.

**Theorem 2.** If $R$ is right $s$-unital ring satisfying $P(0, n, p, q)$, then $R$ is commutative (and conversely).

**Proof.** Consider the ring of type (i). Then
\[
[e_{12} e_{22}^n - e_{22}^n f(e_{12}) e_{22}^q, e_{22}] = e_{12} \neq 0,
\]
for all integers $n \geq 0$, $p \geq 0$, $q \geq 0$ and $f(t) \in t^2 \mathbb{Z}[t]$. Thus, $R$ has no factorsubrings of type (i), and by the dual of Lemma 2, $R$ is $s$-unital. Now, using the same arguments as used in the proof of Theorem 1, we get the required result.

As corollaries to our theorems we have the following results improving [1, Theorem 1.1], [4, Theorem], [5, Theorem 2 (iii)], [6, Theorems 1-4], [7, Theorems 4 & 5], [15, Corollary 2 (3)], [18, Theorem] and [20, Theorem 2 (5)]. Also, note that Theorem 1 generalizes the results proved in [15, Theorem 2] and [16, Theorem 1].

**Corollary 1.** Let $m$, $p$ and $q$ be fixed non-negative integers, and let $R$ be a left $s$-unital ring. If for each $x, y$ in $R$, there exists an integer $s=s(x, y)>1$ such that $[x^m y - x^p y^q x^q, x] = 0$, then $R$ is commutative (and conversely).
Corollary 2. Let $n, p, q$ be fixed non-negative integers, and let $R$ be a right $s$-unital ring. If for each $x, y$ in $R$, there exists an integer $s = s(x, y) > 1$ such that $[yx^n - x^p y^e x^q, x] = 0$, then $R$ is commutative (and conversely).

Theorem 3. Let $R$ be a left $s$-unital ring satisfying $P^*(m, 0, p, q)$ and $(I - N(R))$. Then $R$ is commutative (and conversely).

Proof. It is easy to see that the arguments given in the first paragraph of the proof of Theorem 1 are still valid in the present situation. So we assume henceforth that $R$ has unity 1 and no rings of type (i) satisfy the condition $P^*(m, 0, p, q)$. Also, if $R$ is a ring of type (ii), then choose $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$

$(\sigma(a) \neq a)$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, to get

$[x^m y - x^p f(y) x^q, x] = - x^m [x, y] = - a^m (a - \sigma(a)) y \neq 0$,

for every $f(t) \in I^2 Z[t]$. Thus no rings of type (ii) satisfy $P^*(m, 0, p, q)$. Since $N(R) \subseteq Z(R)$ by Lemma 4, it is straightforward to see that $R$ satisfies (C). Hence, in view of Lemma 1, $R$ is commutative.

The following theorem can also be proved on the same lines as above, employing necessary variations.

Theorem 4. Let $R$ be a right $s$-unital ring satisfying $P^*(0, n, p, q)$ and $(I - N(R))$. Then $R$ is commutative (and conversely).

As an immediate consequence of the above theorems, we obtain the following results improving [2, Theorem 2], [5, Theorem 2 (iii)], [15, Corollary 2 (2)] and [20, Theorem 2 (4)].

Corollary 3. Let $R$ be a left $s$-unital ring. Suppose that for each $x, y$ in $R$, there exist integers $m \geq 0, p \geq 0, q \geq 0$ and $s > 1$ such that $[x^m y - x^p y^e x^q, x] = 0$ and for each $x$ in $R$ either $x$ is central or there exists $f(t) \in I^2 Z[t]$ such that $x - f(x) \in N(R)$. Then $R$ is commutative (and conversely).

Corollary 4. Let $R$ be a right $s$-unital ring. Suppose that for each $x, y$ in $R$, there exist integers $n \geq 0, p \geq 0, q \geq 0$ and $s > 1$ such that $[yx^n - x^p y^e x^q, x] = 0$ and for each $x$ in $R$ either $x$ is central or there exists $f(t) \in I^2 Z[t]$ such that $x - f(x) \in N(R)$. Then $R$ is commutative (and conversely).

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