A NOTE ON AN INTERIOR SOLUTION FOR A FLUID SPHERE OF
CONSTANT GRAVITATIONAL MASS DENSITY IN GENERAL
RELATIVITY.

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In this paper it has been obtained the solutions of the field equations for
a fluid sphere of constant gravitational mass density.

I. Introduction

Schwarzschild interior solution of the field equations for a fluid sphere
of constant density $\rho$ is quite well known and is of great importance in general
relativity. Recently, in three subsequent papers (1972, 1973, 1976), the author
has studied planetary structures in general relativity, in general, and the in­
ternal structure of Mars in particular. So far as known to the author, a solution
of the field equations for a fluid sphere (which is in hydrostatic equilibrium)
of constant gravitational mass density $\rho c^2 + 3P = H$ (1935) has not been at­
ttempted. Gravitational attraction of matter is governed by this expression
(rather than $p$). The physical significance of the parameter $HK/2g^2$ has been
pointed out.

II. The Equations of Equilibrium

In the case of spherical symmetry, the time-independent metric in $(r, \theta, \phi)$-
coordinate system is given by (Moller, 1952)

$$a(r) \, dr^2 + r^2 \, d\Omega^2 - b(r) \, c^2 \, dt^2,$$

(1)

where $d\Omega^2 = (d\theta^2 + \sin^2 \theta \, d\phi^2)$. The gravitational equations, needed for our
purpose, reduce to

$$\frac{1}{ar} \left( \frac{1}{b} \frac{db}{dr} + \frac{1}{r} \right) - \frac{1}{r^2} + \lambda = KP$$

(2)

and

$$\frac{1}{ar} \left( \frac{1}{a} \frac{da}{dr} - \frac{1}{r} \right) + \frac{1}{r^2} - \lambda = K\rho c^2,$$

(3)

where symbols have their usual significance. In the general theory, the hydro­
static equilibrium equation becomes
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\[ \frac{dP}{dr} + (P + \varepsilon) \frac{1}{2b} \frac{db}{dr} = 0, \]  

(4)

where \( P \) is the pressure and \( \varepsilon = \rho c^2 \) is the energy density written as an equivalent to mass density \( \rho \) times the square of the speed of light \( c^2 \).

III. The Desired Solutions

For a fluid sphere of constant gravitational mass density, our assumed relation, connecting pressure with the energy density, is

\[ \rho c^2 + 3P = H = \text{constant}. \]  

(5)

Substituting (5) into (4), we obtain a solution

\[ H - 2P = ab, \]  

(6)

where \( a \) is the constant of integration. Adding (2) and (3), we have

\[ \frac{1}{b} \frac{db}{dr} + \frac{1}{a} \frac{db}{dr} = Kar (P + \rho c^2), \]  

(7)

or, in view of equations (5) and (6), this becomes

\[ \frac{d}{dr} (ab) = Kar (ab)^2, \]  

(7')

the integration of which yields

\[ ab = \frac{2\beta}{2 - K a \delta^2 r^2} = \frac{\beta}{1 - \delta^2 r^2}, \]  

(8)

where \( \beta \) is another constant of integration and \( \delta = (K a \beta / 2)^{\frac{1}{2}} \). Making use of equations (6) and (8) in (2), we obtain

\[ (1 - \delta^2 r^2) r \frac{db}{dr} + b = \beta + \frac{\beta}{2} (KH - 2\lambda) r^2. \]  

(9)

Solving this linear differential equation, we obtain

\[ b(r) = \beta \left( 1 + L + \frac{L}{\delta r} (1 - \delta^2 r^2)\frac{3}{4} \sin^{-1} \delta r \right) + \frac{M}{r} (1 - \delta^2 r^2)^{\frac{3}{4}}, \]  

(10)

where

\[ L = \frac{1}{2\delta^3} (KH - 2\lambda); \ M = \text{constant}. \]  

(11)

Since \( b(r) \) remains finite at the center \( r=0 \) of the fluid sphere, \( M \) must vanish. Thus equation (10), simply, reduces to
Substituting (12) into (8), we get
\[ a(r) = \left\{ (1 + L) \left( 1 - \delta^2 r^2 \right) - L \left( \delta r \right)^{-1} \left( 1 - \delta^2 r^2 \right)^{\frac{3}{2}} \sin^{-1} \delta r \right\}^{-1} . \] (13)

In view of the above, the product of \( K \) and the pressure can be expressed as
\[ KP = \lambda - \delta^2 + L \delta \left( 1 - \delta^2 r^2 \right)^{\frac{3}{2}} \sin^{-1} \delta r . \] (14)

The central values of \( KP \) and \( (Kp) c^2 \) are given by
\[ (KP)_{r=0} = \lambda + (L - 1) \delta^2 = 0.5 KH - \delta^2 , \] (15)
and
\[ (Kp)_{c=0} c^2 = 3\delta^2 - 0.5 KH \] (16)
respectively. These equations make it clear that \( KH \) must satisfy the inequality
\[ \delta^2 \leq \frac{KH}{2} \leq 3\delta^2 , \] (17)
so that \( P_0 \) and \( p_0 \) remain positive. One may verify that \( dP/dr \) is negative, the physical meaning of which is that as the radius increases, the pressure decreases and attains a zero value at the surface boundary \( r = r_1 \) of the sphere, where \( x_1 = ar_1 \) satisfies
\[ (\delta^2 - \lambda) x_1 = \frac{1}{2} (KH - 2\lambda) \left( 1 - x_1^2 \right)^{\frac{3}{2}} \sin^{-1} x_1 , \] (18)
and we find that it lies between 0 and 1. At this point we may replace the interior solution by the Schwarzschild exterior solution for which
\[ b(r) = \frac{1}{a(r)} = 1 - \frac{2m}{r} - \frac{1}{3} r . \] (19)

For the exterior solution \( a(r) \) and \( b(r) \), \( P \) and \( pc^2 \) should be continuous everywhere and \( ab = 1 \), therefore, from (8), we find that
\[ \beta = 1 - \delta^2 r_1^2 . \] (20)

Also, from equations (6) and (19), we find
\[ \frac{H}{a} = \frac{HK\beta}{2\delta^2} = 1 - \frac{2m}{r_1^2} - \frac{1}{3} \lambda r_1^2 = \frac{HK}{2\delta^2} \left( 1 - \delta^2 r_1^2 \right) . \] (21)
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We may note that \( \lambda \) (cosmical constant) has been taken into consideration for the sake of completeness, but in actual situation where the formulae are to be applied we may ignore the same (if not actually zero).

We define the value of the parameter \( \frac{HK}{2\delta^2} \), in terms of the central pressure and central density, as follows:

\[
\frac{HK}{2\delta^2} = \frac{\rho_0 c^2 + 3 P_0}{\rho_0 c^2 + P_0}.
\]

If we put \( \lambda = 0 \) in (21), then

\[
\frac{2m}{r_1} = 1 - \left( \frac{HK}{2\delta^2} (1 - \delta^2 r_1^2) \right).
\]

Hence, in order that \( m \) remains positive,

\[
\frac{HK}{2\delta^2} (1 - \delta^2 r_1^2) < 1,
\]

which is a strict inequality so as to satisfy the Schwarzschild exterior solution.

REFERENCES


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ÖZET

Bu çalışmada gravitasyonel kütle yoğunluğu sabit olan bir akışkan küre için alan denklemlerinin çözümleri elde edilmektedir.