ACZEL has shown that if \( S \) is a sub-semigroup of the group \( G \), such that
\[
x \in G \implies x \in S \text{ or } x' \in S \quad \text{(where } x' \text{ is the inverse of } x)\]
then every homomorphism \( f \) of \( S \) into a group \( H \) can be extended in a unique way to a homomorphism \( g \) of \( G \) into \( H \) such that \( f \) and \( g \) coincide over \( S \) and
\[
g(xy) = g(x)g(y)
\]
for all \( x, y \in G \).

The object of this paper is to evolve suitable techniques to extend an homomorphism \( f \) from a subgroup \( S \) of the group \( G \) into a group \( H \) to a group homomorphism \( g \) of \( G \) into \( H \), such that \( f \) and \( g \) coincide on \( S \) and the functional equation (f), or some analogue of this equation, hold.

1. Introduction. While studying some extensions of certain homomorphisms of subsemigroups to homomorphisms of groups, J. ACZEL et al.\(^{[3]}\) proved the following theorem:

**Theorem 1.** Let \( S \) be a subsemigroup of a group \( G \) such that for each element \( x \) of \( G \), different from the identity element of \( G \), either \( x \in S \), or \( x' \in S \) (or both) where \( x' \) denotes the inverse of \( x \). Then, every homomorphism \( f \) of \( S \) into a group \( H \) can be extended in a unique way to a homomorphism \( g \) of \( G \) into \( H \) such that

\[
g(x) = f(x), \quad \forall x \in S
\]
and
\[
g(xy) = g(x)g(y), \quad x \in G, y \in G.
\]
It is clear that if $S$ is a subgroup of the group $G$, then $x \in S \iff x' \in S$. Hence the condition that for each element $x$ of $G$, different from the identity element of $G$, either $x \in S$ or $x' \in S$ (or both) no longer holds. Accordingly, the methods developed in [3] do not serve our purpose, if we want to extend the homomorphism $f$ of the subgroup $S$ into $H$ to a group homomorphism $g$ of $G$ into $H$ such that $g(x) = f(x)$ for all $x \in S$ and $g(xy) = g(x)g(y)$ for all $x, y \in G$.

The object of this paper is to study the problem of extending homomorphisms of subgroups to homomorphisms of groups. Not every subgroup homomorphism can be extended to a group homomorphism. However, in certain cases, it is possible to extend a group homomorphism to a group homomorphism. To demonstrate this, the authors have restricted to THIELMAN's functional equations.

2. The Sets $A_n$, $I_n$ and $A^*_n$. Let $R = (-\infty, \infty)$ and $E \subset R$. Following CHEVALLEY [5], a subset $E$ of $R$ is said to be stable with respect to the binary law of composition $\tau^n$ if $x \in E$, $y \in E \implies xy \in E$. Let us define the disjoint subsets $A_n$ and $I_n$ of $R$ as follows:

$$A_n = \left\{ x \in R : x > -\frac{1}{n} \right\}, \quad n > 0,$$

$$I_n = \left\{ x \in R : x < -\frac{1}{n} \right\}, \quad n > 0.$$

If $\tau^n$ denotes the ordinary arithmetic addition, then $A_n$ is not stable. For example, if $n = 1$, $x = -\frac{7}{8}$, $y = -\frac{3}{8}$, then $x + y = -\frac{10}{8} < -1$ and thus $x + y \notin A_1$. However, if we consider the family of binary operations $\tau^n$, $n > 0$, defined by

$$(A) \quad x \tau^n y = x + y + nxy,$$

where, on the right hand side of $(A)$, we have ordinary arithmetic addition and multiplication, then $(A_n, \tau^n)$ is a commutative group with real number
0 as the identity element. But \( \Gamma_n \) is not stable with respect to the binary operation \( \tau^n \) because \( x < - \frac{1}{n}, \ y < - \frac{1}{n} \) implies \( x\tau^n y > - \frac{1}{n}, \ n > 0. \)

Let \( A_n^* = A_n \cup \Gamma_n, \ n > 0. \) Obviously, \( A_n^* = R - \left\{ - \frac{1}{n} \right\} \) and \( (A_n^*, \tau^n) \) is a commutative group of which \( (A_n, \tau^n) \) is a proper subgroup. Clearly, \( \Gamma_n \) also denotes the set of those points which belong to the group \( (A_n^*, \tau^n) \) but not to the subgroup \( (A_n, \tau^n) \). The following can be easily derived by making use of (A).

(a) Denoting by \( x' \), the inverse of \( x \in (A_n, \tau^n) \), it can be easily seen that \( x' = - \frac{x}{1 + nx} \) and further \( x \in \Gamma_n \iff x' \in \Gamma_n. \)

(b) \( x \in (A_n, \tau^n), \ y \in (A_n, \tau^n) \Rightarrow x\tau^n y \in (A_n, \tau^n) \) and \( x\tau^n y' \in (A_n, \tau^n). \)

(c) \( x \in \Gamma_n, \ y \in (A_n, \tau^n) \Rightarrow x\tau^n y \in \Gamma_n \) and \( x\tau^n y' \in \Gamma_n. \)

(d) \( x \in (A_n, \tau^n), \ y \in \Gamma_n \Rightarrow x\tau^n y \in \Gamma_n \) and \( x\tau^n y' \in \Gamma_n. \)

(e) \( x \in \Gamma_n, \ y \in \Gamma_n \Rightarrow x\tau^n y \in (A_n, \tau^n) \) and \( x\tau^n y' \in (A_n, \tau^n). \)

(f) \( x \in A_n \iff \left(-x - \frac{2}{n}\right) \in \Gamma_n. \)

\[
\begin{align*}
(x\tau^n y) - \frac{2}{n} &= x\tau^n \left(-y - \frac{2}{n}\right), & \text{if } x \in (A_n, \tau^n), \ y \in \Gamma_n, \\
&= \left(-x - \frac{2}{n}\right) \tau^n y, & \text{if } x \in \Gamma_n, \ y \in (A_n, \tau^n).
\end{align*}
\]

(g) \( x\tau^n y = \left(-x - \frac{2}{n}\right) \tau^n y \), \( x \in (A_n^*, \tau^n), \ y \in (A_n^*, \tau^n). \)

From the above observations, it is clear that for all \( x \in (A_n^*, \tau^n), \ y \in (A_n^*, \tau^n), \) the elements \( x\tau^n y \) and \( x\tau^n y' \) belong simultaneously either to \( (A_n, \tau^n) \) or to the set \( \Gamma_n. \)

\[ f_n(x + y + nxy) = g_n(x) + h_n(y), \quad x \in A_n, \ y \in A_n \]

and

\[ f_n(x + y + nxy) = g_n(x) h_n(y), \quad x \in A_n, \ y \in A_n, \]

where \( f_n, g_n \) and \( h_n \) are real-valued continuous functions with domain \( A_n \). We shall consider the following more general functional equation

\[ f_n(x + y + nxy) = g_n(x) K(y), \]

in which the functions \( f_n, g_n \) and \( h_n \) are defined on the subgroup \((A_n, \tau^n)\) and they take their values in an arbitrary group \( \mathcal{E} \) which contains no zero element, that is, there does not exist any element \( b \in \mathcal{E} \) such that

\[ bx = xb = b, \quad \text{for all } x \in \mathcal{E}. \]

It should be noted that on the right hand side of (3), \( g_n(x) h_n(y) \) is to be computed in accordance with the group operation in \( \mathcal{E} \).

Since every group is a semigroup, following the method of J. ACZÉL [2], the theorem given below can be easily proved:

**Theorem 2.** The most general solutions of (3) among the functions \( f_n, g_n, h_n \) mapping the commutative subgroup \((A_n, \tau^n)\) into an arbitrary group \( \mathcal{E} \), containing no zero element, are given by

\[ f_n(x) = g_n(0) k_n(x) h_n(0), \quad g_n(x) = g_n(0) k_n(x), \quad h_n(x) = k_n(x) h_n(0), \quad x \in (A_n, \tau^n), \]

where \( k_n \) is a homomorphism of \((A_n, \tau^n)\) into \( \mathcal{E} \); i.e.,

\[ k_n(x \tau^n y) = k_n(x) h_n(y), \quad x, y \in (A_n, \tau^n). \]

It may be noted that in the above theorem, it is not assumed that \( \mathcal{E} \) is an abelian group. Also, in (3), \( f_n, g_n \) and \( h_n \) are just functions (not necessarily ho-
momorphisms) and in (4), \( g_n(0) \) and \( h_n(0) \) may be assigned arbitrary values in \( \mathcal{E} \).

4. Extension Theorems Concerning (3). In § 3, we have shown that the functional equation (3) admits of solutions of the form (4). Now, our object is to extend these solutions. Our method will be to obtain an extension \( K_n : (\Delta_n^*, \tau^n) \to \mathcal{E} \) of the subgroup homomorphism \( k_n : (\Delta_n, \tau^n) \to \mathcal{E} \) satisfying (5) and such that

\[
K_n(x \tau^n y) = K_n(x) K_n(y), \quad \text{for all } x, y \in (\Delta_n^*, \tau^n)
\]

and

\[
K_n(x) = k_n(x), \quad \text{for all } x \in (\Delta_n, \tau^n)
\]

and then define the extensions \( F_n, G_n \) and \( H_n \) of \( f_n, g_n \) and \( h_n \) respectively such that

\[
F_n(x \tau^n y) = G_n(x) H_n(y), \quad \text{for all } x, y \in (\Delta_n^*, \tau^n)
\]

and

\[
F_n(x) = f_n(x), \quad G_n(x) = g_n(x), \quad H_n(x) = h_n(x), \quad \forall x \in (\Delta_n, \tau^n).
\]

Theorem 3. Let \( k_n \) be a subgroup homomorphism of \( (\Delta_n, \tau^n) \) into the group \( \mathcal{E} \). Then, to each \( \lambda \in \mathcal{E} \) with \( \lambda \lambda = e \), the identity element of \( \mathcal{E} \), there exists a group homomorphism \( K_{n, \lambda} \) of \( (\Delta_n^*, \tau^n) \) into \( \mathcal{E} \) such that \( K_{n, \lambda} \) is an extension of \( k_n \).

Proof. Define \( K_{n, \lambda} \) as follows:

\[
K_{n, \lambda}(x) = \begin{cases} 
  k_n(x) & \text{if } x \in (\Delta_n, \tau^n), \\
  \lambda k_n \left( -x + \frac{2}{n} \right) & \text{if } x \in \Gamma_n,
\end{cases}
\]

where \( \lambda \in \mathcal{E} \) commutes with each element of the range of \( k_n \) and further
\[ \lambda \lambda = e, \text{ the identity element of } \mathcal{E}. \]

Since \( \lambda \) commutes with each element of the range of \( k_n \), therefore,

\[ \lambda k_n(x) = k_n(x) \lambda, \quad \text{for all } x \in (A_n, \tau^n). \tag{12} \]

Also, we know that \( x \in \Gamma_n \Rightarrow \left( -x - \frac{2}{n} \right) \in A_n \) so that \( k_n \left( -x - \frac{2}{n} \right) \) belong to the range of \( k_n \) and consequently

\[ \lambda k_n \left( -x - \frac{2}{n} \right) = k_n \left( -x - \frac{2}{n} \right) \lambda, \quad \text{for all } x \in \Gamma_n. \tag{13} \]

We discuss the following four cases:

**Case (i).** \( x \in (A_n, \tau^n), \ y \in (A_n, \tau^n). \)

In this case, \( x \tau^n y \in (A_n, \tau^n) \) and thus \( x \tau^n y \in (A_n, \tau^n) \). Hence \( K_{n,\lambda} \) satisfies (6) obviously.

**Case (ii).** \( x \in (A_n, \tau^n), \ y \in \Gamma_n. \)

In this case, because of (d), \( x \tau^n y \in \Gamma_n \). Hence, we have

\[
K_{n,\lambda}(x \tau^n y) \overset{(10)}{=} \lambda k_n \left[ - (x \tau^n y) - \frac{2}{n} \right] \overset{(g)}{=} \lambda k_n \left[ x \tau^n \left( -y - \frac{2}{n} \right) \right]
\]

\[
\overset{(5)}{=} \lambda k_n(x) \overset{(12)}{=} k_n(x) \lambda k_n \left( -y - \frac{2}{n} \right) \overset{(10)}{=} K_{n,\lambda}(x) K_{n,\lambda}(y).
\]

**Case (iii).** \( x \in \Gamma_n, \ y \in (A_n, \tau^n). \)

The proof is similar to that of the case (ii).

**Case (iv).** \( x \in \Gamma_n, \ y \in \Gamma_n. \)

In this case, by (e), \( x \tau^n y \in (A_n, \tau^n) \). Hence, we have
Thus, we have proved that $K_{n, \lambda}$ satisfies (6). The fact that (7) holds is obvious from (10). This completes the proof of the theorem.

From (10), it is clear that if $\mathcal{E}$ contains at least one element $\lambda$, different from $e$, such that $\lambda$ satisfies (11) and (12), then the extension $K_{n, \lambda}$ of $k_n$ is not unique.

Theorem 4. Every group homomorphism $K_n: (A_n^*, \tau^n) \to \mathcal{E}$, which is an extension of the subgroup homomorphism $h_n: (A_n, \tau^n) \to \mathcal{E}$, is of the form (10) with $\lambda$ satisfying (11) and (12).

Proof. Since $K_n$ is an extension of $k_n$, therefore, we have (7). Now we determine the form of $K_n(x)$ for $x \in \Gamma_n$. Let $x$ be any element of $\Gamma_n$. Then, for all $z \in \Gamma_n$, we have

$$K_n(x) = K_n(x \tau^n z \tau^n x') \overset{(6)}{=} K_n(x \tau^n z) K_n(z') \overset{(7)}{=} k_n(x \tau^n z) K_n(z')$$

$$\overset{(h)}{=} k_n \left[ \left( -x - \frac{2}{n} \right) \tau^n \left( -z - \frac{2}{n} \right) \right] K_n(z') \overset{(5)}{=} k_n \left( -x - \frac{2}{n} \right) K_n(z')$$

$$\overset{(10)}{=} K_{n, \lambda}(x) K_{n, \lambda}(z').$$
\((6)\) \[ k_n \left( -x - \frac{2}{n} \right) K_n \left( -z - \frac{2}{n} \right) K_n(z') = k_n \left( -x - \frac{2}{n} \right) \]

\[ K_n \left[ \left( -z - \frac{2}{n} \right) \tau^n z' \right]. \]

But

\[ \left( -z - \frac{2}{n} \right) \tau^n z' = \left( -z - \frac{2}{n} \right) \tau^n \left( \frac{-z}{1 + nz} \right) = \left( -z - \frac{2}{n} \right) \]

\[ + \left( \frac{-z}{1 + nz} \right) + n \left( -z - \frac{2}{n} \right) \left( \frac{-z}{1 + nz} \right) = -\frac{2}{n}. \]

Hence

\[ K_n(x) = k_n \left( -x - \frac{2}{n} \right) \lambda, \quad \text{where } \lambda = K_n \left( -\frac{2}{n} \right). \]

Similarly,

\[ K_n(x) = K_n(z') \tau^n \tau^n x = K_n(z') K_n(\tau^n x) = K_n(z') k_n(\tau^n x) \]

\[(h)\]

\[ K_n(z') k_n \left[ \left( -z - \frac{2}{n} \right) \tau^n \left( -x - \frac{2}{n} \right) \right] \quad \text{(5)} \]

\[ K_n(z') k_n \left( -z - \frac{2}{n} \right) \]

\[ = k_n \left[ z' \tau^n \left( -z - \frac{2}{n} \right) \right] k_n \left( -x - \frac{2}{n} \right) = K_n \left( -\frac{2}{n} \right) k_n \left( -x - \frac{2}{n} \right) \]

\[ = \lambda k_n \left( -x - \frac{2}{n} \right). \]

Thus

\[ K_n(x) = k_n \left( -x - \frac{2}{n} \right) \lambda = \lambda k_n \left( -x - \frac{2}{n} \right), \quad x \in \Gamma_n, \lambda = K_n \left( -\frac{2}{n} \right). \]
But
\[ K_n\left( \frac{-2}{n} \right) = K_n\left( \frac{-2}{n} \right) = K_n\left( \frac{-2}{n} \right) = \lambda. \]

Actual computation gives \( \left( \frac{-2}{n} \right) = 0 \), the identity element of \((A^*, \tau^n)\). Since \( K_n \) is a group homomorphism, therefore, we must have \( K_n(0) = e \), the identity element of \( S \). Thus \( \lambda = e \), which is (11). Since there may exist more than one \( \lambda \) satisfying (11), writing \( K_n \) as \( K_{n, \lambda} \), the required conclusion follows.

Remark. Let us define mappings \( \phi_n : A^*_n = R - \{0\} \) as
\[ \phi_n(x) = nx + 1, \quad x \in A^*_n. \]
Then, it can be easily verified that
\[ \phi_n(x \tau^n y) = \phi_n(x) \phi_n(y), \quad \text{for all } x \text{ and } y \text{ in } A^*_n. \]

Also, \( \phi_n(x) > 0 \) if and only if \( x \in A_n \). Since \( \phi_n \) also is a bijection, therefore, from the above observations, it follows that \( \phi_n \) induces an isomorphism between \((A_n, \tau^n)\) and the group \((V_0, \cdot)\) where \( V_0 = (0, \infty) \). Then, (5) can be written in the form
\[ k_n(\phi_n^{-1}(uv)) = k_n(\phi_n^{-1}(u)) k_n(\phi_n^{-1}(v)), \quad u > 0, \ v > 0. \]
If we write
\[ \psi_n(u) = k_n(\phi_n^{-1}(u)), \quad u > 0, \]
then
\[ (B) \quad \psi_n(uv) = \psi_n(u) \psi_n(v), \quad u > 0, \ v > 0. \]

The main advantage in dealing with (B) is that the argument of \( \psi_n \) on the L.H.S. of (B) is also independent of \( n \) as compared with that of \( k_n \) in (5).
If \( K_n \) is an extension of \( k_n \), then by the above reasoning, the mapping \( \Psi_n : R - \{0\} \to S \) defined by
\[ \Psi'_n(u) = K_n(\phi_n^{-1}(u)), \quad u \neq 0, \]

is a homomorphism of \( R - \{0\} \) into \( \mathscr{S} \) and it extends \( \psi_n \). If we can find the extension \( \Psi'_n \), then with the aid of \( \psi_n \), we can also find the corresponding form of \( K_n \). However, the method explained in the proof of theorem 4 readily gives us the forms of extensions if they exist and theorem 3 ensures that they are indeed the extensions of \( k_n \).

Now we give an extension theorem concerning the functional equation (3).

**Theorem 5.** If the functions \( f_n, g_n, h_n \) defined on the subgroup \( (A_n, \tau^n) \) satisfy the functional equation (3) with their values lying in a group \( \mathscr{S} \) containing no zero element, then the functions \( F_n, G_n, H_n \) defined on the group \( (A'_n, \tau^n) \), with their values in \( \mathscr{S} \), by

\[
F_n(x) = g_n(0) K_n(x) h_n(0), \quad G_n(x) = g_n(0) K_n(x), \quad H_n(x) = K_n(x) h_n(0),
\]

where \( K_n \) is an extension of the subgroup homomorphism \( k_n : (A_n, \tau^n) \rightarrow \mathscr{S} \) satisfying (5), are the extensions of \( f_n, g_n, h_n \) respectively in the sense that they satisfy (8) and (9).

**Proof.** We have

\[
F_n(x \tau^n y) = g_n(0) K_n(x \tau^n y) h_n(0) = g_n(0) K_n(x) K_n(y) h_n(0) = G_n(x) H_n(y).
\]

This proves the theorem.

In theorem 5, we have assumed that \( \mathscr{S} \) contains no zero element. If \( \mathscr{S} \) contains a zero element, say \( b \), then

\[
f_n(x) = b, \quad g_n(x) \text{ arbitrary}, \quad h_n(x) = b,
\]

\[
f_n(x) = b, \quad g_n(x) = b, \quad h_n(x) \text{ arbitrary},
\]

are also (trivial) solutions of (3). The extensions of these solutions are not of any importance and we shall not consider them.
For a fixed \( \lambda \), let us write \( F_n, G_n \) and \( H_n \) as \( F_{n, \lambda}, G_{n, \lambda} \) and \( H_{n, \lambda} \) respectively. Then (10) and (14) give

\[
\begin{align*}
F_{n, \lambda}(x) &= g_n(0) h_n(x) h_n(0), & x \in \mathcal{A}_n, \\
&= g_n(0) \lambda k_n \left( -x - \frac{2}{n} \right) h_n(0), & x \in \Gamma_n, \\
G_{n, \lambda}(x) &= g_n(0) h_n(x), & x \in \mathcal{A}_n, \\
&= g_n(0) \lambda k_n \left( -x - \frac{2}{n} \right), & x \in \Gamma_n, \\
H_{n, \lambda}(x) &= h_n(x) h_n(0), & x \in \mathcal{A}_n, \\
&= \lambda k_n \left( -x - \frac{2}{n} \right) h_n(0), & x \in \Gamma_n.
\end{align*}
\]

(15)

where \( \lambda \) satisfies (11) and (12). Also from (4), we have

\[
(16) \quad h_n(x) = [g_n(0)]' f_n(x) [h_n(0)]' = [g_n(0)]' g_n(x) = h_n(x) [h_n(0)]', \quad x \in (\mathcal{A}_n; \tau^\mu).
\]

Hence, in terms of \( g_n \), (12) and (13) reduce to the form

\[
(17) \quad \lambda [g_n(0)]' g_n(x) = [g_n(0)]' g_n(x) \lambda, \quad x \in \mathcal{A}_n,
\]

and

\[
(18) \quad \lambda [g_n(0)]' g_n \left( -x - \frac{2}{n} \right) = [g_n(0)]' g_n \left( -x - \frac{2}{n} \right) \lambda, \quad x \in \Gamma_n.
\]

Similarly, in terms of \( h_n \), (12) and (13) take the form

\[
(19) \quad \lambda h_n(x) [h_n(0)]' = h_n(x) [h_n(0)]' \lambda, \quad x \in \mathcal{A}_n,
\]

and

\[
(20) \quad \lambda h_n \left( -x - \frac{2}{n} \right) [h_n(0)]' = h_n \left( -x - \frac{2}{n} \right) [h_n(0)]' \lambda, \quad x \in \Gamma_n.
\]

Also (15) reduces to
Now, we prove the following theorem:

**Theorem 6.** If the functions $f_n, g_n, h_n$ defined on the subgroup $(A_n, \tau^n)$, satisfy the functional equation (3) with their values lying in a group $\mathcal{E}$ containing no zero element, then for each $\lambda \in \mathcal{E}$ satisfying (11), the functions $F_{n,\lambda}$, $G_{n,\lambda}$ and $H_{n,\lambda}$ defined by (21) are extensions of $f_n, g_n$ and $h_n$ respectively in the sense that they satisfy (8) and (9).

**Proof.** As in the proof of theorem 3, we discuss the same four cases.

**Case (i).** $x \in (A_n, \tau^n), \ y \in (A_n, \tau^n)$. Then,$$
F_{n,\lambda}(x \tau^n y) = f_n(x \tau^n y) = g_n(x) h_n(y) = G_{n,\lambda}(x) H_{n,\lambda}(y).
$$

**Case (ii).** $x \in (A_n, \tau^n), \ y \in \Gamma_n$. Then

$$
F_{n,\lambda}(x \tau^n y) \overset{(21)}{=} g_n(0) \lambda [g_n(0)]' f_n \left[ - (x \tau^n y) - \frac{2}{n} \right]
$$

$$
= g_n(0) \lambda [g_n(0)]' f_n \left[ x \tau^n \left( - y - \frac{2}{n} \right) \right]
$$

$$
\overset{(3)}{=} g_n(0) \lambda [g_n(0)]' g_n(x) h_n \left( - y - \frac{2}{n} \right)
$$
Case (iii).  \( x \in \Gamma_n, \ y \in (A_n, \tau^a) \). Then

\[
F_{n,\lambda}(x\tau^a y) \overset{(21)}{=} g_n(0) \lambda [g_n(0)]^* f_n \left( -\frac{2}{n} \right) \tau^a y
\]

\[
= g_n(0) \lambda [g_n(0)]^* f_n \left( -x - \frac{2}{n} \right) \tau^a y
\]

\[
\overset{(3)}{=} g_n(0) \lambda [g_n(0)]^* g_n \left( -x - \frac{2}{n} \right) h_n(y)
\]

\[
\overset{(21)}{=} G_{n,\lambda}(x) H_{n,\lambda}(y).
\]

Case (iv).  \( x \in \Gamma_n, \ y \in \Gamma_n \). Then,

\[
F_{n,\lambda}(x\tau^a y) = f_n(x\tau^a y) = f_n \left( -x - \frac{2}{n} \right) \tau^a \left( -y - \frac{2}{n} \right)
\]

\[
= g_n(0) \left[ g_n(0) \right] \lambda h_n \left( -y - \frac{2}{n} \right)
\]

\[
= g_n(0) \lambda [g_n(0)]^* g_n \left( -x - \frac{2}{n} \right) \lambda h_n \left( -y - \frac{2}{n} \right)
\]

\[
\overset{(18)}{=} G_{n,\lambda}(x) H_{n,\lambda}(y).
\]

This completes the proof of the theorem.

We hope to discuss some more methods of extending subgroup homomorphisms elsewhere.
REFERENCES


ÖZET

ACZEL, $x'$, grup işlemine göre $x$ elementinin tersini göstermek üzere,

$$x \in G \Rightarrow x \in S \text{ veya } x' \in S$$

koşulunu sağlayan $G$ grubunun bir $S$ alt-semigrubu üzerinde tanımlanmış ve bu $S$ semigrubunu bir $H$ grubunun içine taşır eden bir $f$ homomorfizması, her $x, y \in G$ için

$$f(xy) = f(x)f(y)$$

olacak şekilde $G$ grubundan $H$ grubuna bir $g$ homomorfizmasına $f$ ve $g S$ üzerinde çakışacak tarzda tek bir şekilde genişletebileceği göstermiştir.

Bu araştırmada ise $S$ nin, $G$ grubunun bir alt grubu olması halinde yukarıdakine benzer teoremler elde etmeye uğraştıkları; $S$ nin bir semigrup değil, bir grup olmadığı ACZEL'in yönteminin bu hale uygulanmasını önlemektedir. Ayrıca ($f$) fonksiyonel denkleminine bazı başka şekiller verilmesi de öngörülmüştür.

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