SUBCLASSES OF UNIVALENT MEROMORPHIC FUNCTIONS

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Let \( M_n \) be the classes of regular functions \( f(z) = z - a_0 + a_1 z + \ldots \) defined in the annulus \( 0 < |z| < 1 \) and satisfying \( \text{Re} \frac{f^{n+1}(z)}{f^n(z)} > 0 \) (\( n = 0, 1, 2, \ldots \)), where \( f f(z) = f(z) * g(z) = \frac{(1-z)g(z)}{z} \), \( l^n f(z) = l^n f(z) = I l^{n-1} f(z) \) and \( * \) is the Hadamard convolution. We denote by \( R_n (x, \beta) \) the set of all functions \( f(z) = z - a_0 + a_1 z + \ldots \) such that

\[
\text{Re} \left\{ (z + \beta) \frac{l^{n+1} f(z)}{l^n f(z)} - a \frac{l^{n+1} f(z)}{l^n f(z)} \right\} > 0 \quad (|z| < 1, a > 0, \beta > 0).
\]

It is proved that \( M_{n+1} \subset M_n \) and \( R_n (x, \beta) \subset M_n \). In particular we obtain the radius of \( M_{n+1} \) for the class \( M_n \). Further we consider the integrals of functions in \( M_n \).

1. INTRODUCTION

Let \( f(z) = \frac{1}{z} + a_0 + a_1 z + \ldots \) be regular in the annulus \( 0 < |z| < 1 \). The Hadamard product or the convolution of two functions \( f, g \in \Sigma \) will be denoted by \( f * g \). The convolution has algebraic properties of ordinary multiplication.

Let \( n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \). We define a linear operator \( I^n \) on \( \Sigma \) by

\[
I^0 f(z) = f(z)
I^1 f(z) = I f(z) = -z f'(z)
I^n f(z) = I(I^{n-1} f(z)) = -z (I^{n-1} f(z))'.
\]

This operator is an iterated convolution \([7]\). If \( h(z) = \frac{1}{z} - \frac{z}{(1-z)^2} \), then

\[
I^n f(z) = ((h * h * \ldots * h) * f) (z) = \frac{1}{z} + (-1)^n \sum_{k=1}^{\infty} k^n a_k z^k.
\]

Now we introduce the following classes: Let \( E \) denote the unit disc, \( \{ z : |z| < 1 \} \). We define the classes \( M_n \) of functions \( f \in \Sigma \) and satisfying the condition
For every $n \in N_0$, $M_n$ contains many interesting classes of univalent functions: $M_0$ and $M_1$ are known classes of univalent functions that are meromorphically starlike and convex respectively. Let

$$r_n(z) = (\alpha + \beta) \frac{I^{n+1}f(z)}{I^n f(z)} - \alpha \frac{I^{n+2}f(z)}{I^{n+1} f(z)} \quad (1.1)$$

where $n \in N_0$ and $\alpha, \beta$ are non-negative real numbers. We say that $f \in R_n(\alpha, \beta)$, if $f \in \Sigma$ and

$$\text{Re} r_n(z) > 0 \quad (z \in E, \alpha > 0, \beta > 0, n \in N_0).$$

It is clear that $R_n(0, \beta) = M_n, R_n(-1, 1) = M_{n+1}$.

In Section 2 we shall show that

$$M_{n+1} \subset M_n \quad (n \in N_0). \quad (1.2)$$

Methods used are similar to those in [4]. Since $M_0$ equals $\Sigma^*$ (the class of meromorphically starlike functions) the univalence of members in $M_n$ is a consequence of (1.2). In particular we obtain the radius of $M_{n+1}$ for the class $M_n$.

Note that when $n = 0$, this number $2 - \sqrt{3}$ is called the radius of convexity for the class of meromorphically starlike functions. Next we shall show that $R_n(\alpha, \beta) \subset M_n$. For $n = 0$ it follows that $R_0(\alpha, \beta) \subset \Sigma^*$. This result is a generalization of the result obtained by Bajpai-Mehrok in [1]. In Section 3 we study special elements of $M_n$ which have certain integral representations. Our results are thus generalizations of the results obtained by Goel-Sohi in [2].

2. THE CLASSES $M_n$ AND $R_n(\alpha, \beta)$

**Theorem 1.** $M_{n+1} \subset M_n$ for all $n \in N_0$.

**Proof.** Let $f \in M_{n+1}$. We define $w(z)$ in $E$ by

$$\frac{I^{n+1}f(z)}{I^n f(z)} = \frac{1 - w(z)}{1 + w(z)}. \quad (2.1)$$

Here $w(z)$ is a regular function in $E$ with $w(0) = 0$ and $w(z) \neq -1$ for $z \in E$. Differentiating (2.1) logarithmically we obtain

$$\frac{I^{n+2}f(z)}{I^{n+1} f(z)} = \frac{1 - w(z)}{1 + w(z)} + \frac{2z w'(z)}{(1 - w(z)) (1 + w(z))}. \quad (2.2)$$

Equation (2.1) should yield $|w(z)| < 1$ for all $z \in E$, otherwise by a lemma of Jack [3] there exists $z_0 \in E$ such that $z_0 w'(z_0) = m w(z_0)$, $m \geq 1$, $|w(z_0)| = 1$. Applying this result to (2.2) we get
This is a contradiction to the assumption that \( f \in M_{n+1} \). Hence \( f \in M_n \) when \( n \in \mathbb{N}_0 \).

**Remark.** Since \( M_0 \) equals \( \Sigma^* \) (the class of starlike functions), it follows from the above theorem that all functions in \( M_n \) are univalent.

**Theorem 2.** Let \( f \in M_n \). Then \( \text{Re} \frac{I^{n+2}f(z)}{I^{n+1}f(z)} > 0 \) holds for \( |z| < 2 - \sqrt{3} \).

**Proof.** Let \( p(z) \) be the regular function defined in \( E \) by

\[
\frac{I^{n+1}f(z)}{I^n f(z)} = p(z).
\]

(2.3)

Here \( p(0) = 1 \) and \( \text{Re} \ p(z) > 0 \) in \( E \). Logarithmic differentiation of (2.3) yields

\[
\frac{I^{n+2}f(z)}{I^{n+1}f(z)} = p(z) - \frac{zp'(z)}{p(z)}.
\]

(2.4)

Using the well-known estimates

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2|z|}{1 - |z|}, \quad \text{and} \quad \text{Re} \ p(z) \geq \frac{1 - |z|}{1 + |z|},
\]

we see from (2.4) that

\[
\text{Re} \frac{I^{n+2}f(z)}{I^{n+1}f(z)} \geq \text{Re} \ p(z) \cdot \frac{|z|^2 - 4|z| + 1}{1 - |z|^2}.
\]

(2.5)

Now the right hand side of (2.5) is positive provided \( |z| < 2 - \sqrt{3} \).

**Theorem 3.** \( R_n (\alpha, \beta) \subset M_n \) for all \( n \in \mathbb{N}, \alpha > 0 \) and \( \beta > 0 \).

**Proof.** Suppose \( f \in R_n (\alpha, \beta) \) and

\[
\frac{I^{n+1}f(z)}{I^n f(z)} = \frac{1 - w(z)}{1 + w(z)} \quad (z \in E).
\]

(2.6)

Then \( w(z) \) is regular in \( E \) with \( w(0) = 0, w(z) \neq -1 \). To complete the proof we need to show that \( \text{Re} \ \frac{1 - w(z)}{1 + w(z)} > 0, z \in E \). Taking the logarithmic derivative of both sides of (2.6) we get

\[
\frac{I^{n+2}f(z)}{I^{n+1}f(z)} = \frac{1 - w(z) + 2z w'(z)}{1 + w(z) (1 - w(z) (1 + w(z)))}.
\]

(2.7)

Substituting from (2.6) and (2.7) in (1.1) we obtain
\[ r_n(z) = \beta \frac{1 - w(z)}{1 + w(z)} - \alpha \frac{2zw'(z)}{(1 - w(z))(1 + w(z))}. \]  
(2.8)

The conclusion of the theorem from (2.8) follows as in Theorem 1.

3. SPECIAL ELEMENTS OF \( M_n \).

Theorem 4. Let \( n \in \mathbb{N} \) and \( c > 0 \). If \( f \in M_n \), then

\[ F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) \, dt \]  
(3.1)

also belongs to \( M_n \) for \( F(z) \neq 0 \) in \( 0 < |z| < 1 \).

Proof. Since

\[ z F'(z) = cf(z) - (c + 1) F(z) \]

it can be easily verified that

\[ I^{n+1} F(z) = (c + 1) I^n F(z) - c I^n f(z). \]  
(3.2)

Let \( w(z) \) be the regular function in \( E \) defined by

\[ \frac{I^{n+1} F(z)}{I^n F(z)} = \frac{1 - w(z)}{1 + w(z)} (z \in E). \]  
(3.3)

Obviously \( w(0) = 0 \), \( w(z) \neq -1 \) for \( z \in E \). Logarithmic derivative (3.3) and using (3.2) we obtain

\[ \frac{I^{n+1} f(z)}{I^n f(z)} = \frac{1 - w(z)}{1 + w(z)} \frac{2zw'(z)}{(1 + w(z)) (c + (c + 2)w(z))} \]  
(3.4)

We claim that \( |w(z)| < 1 \) in \( E \), otherwise by a lemma of Jack \([7]\) there exists \( z_0 \in E \) such that \( z_0 w'(z_0) = mw(z_0), \ m \geq 1 \), \( |w(z_0)| = 1 \). Thus at \( z = z_0 \), from (3.4) we see that

\[ \text{Re} \left\{ \frac{I^{n+1} f(z_0)}{I^n f(z_0)} \right\} < 0. \]

This completes the proof of the theorem by contradiction.

We shall use the following lemma due to Goel-Sohi (\([7]\), Theorem 4):

Lemma. If \( w(z) \) is regular in \( E \) and satisfies the conditions \( w(0) = 0 \), \( |w(z)| < 1 \) for \( z \in E \), then

\[ \text{Re} \left\{ \frac{1 - w(z)}{1 + w(z)} \frac{2zw'(z)}{(1 + w(z)) (c + (c + 2)w(z))} \right\} > 0 \]
Theorem 5. Let $F \in M_n$, $c > 0$, and $f(z) = \frac{1}{cz} (z^{c+1} F(z))'$, then $f \in M_n$ for $0 < |z| < \sqrt{\frac{c}{c+2}}$. The result is sharp.

Proof. Since $F \in M_n$, there exists a function $w(z)$ regular in $E$ with $w(0) = 0$, $|w(z)| < 1$ such that

$$\frac{I^{n+1} F(z)}{I^n F(z)} = \frac{1 - w(z)}{1 + w(z)}.$$ 

We find (3.4) from the Theorem 4. It follows that from lemma

$$\text{Re} \frac{I^{n+1} f(z)}{I^n f(z)} > 0$$

for $0 < |z| < \sqrt{\frac{c}{c+2}}$.

Let $F(z) = \frac{1}{z} + 2 + z$ and $z_0 = \sqrt{\frac{c}{c+2}}$. Then $I^n F(z) = \frac{1}{z} + (-1)^n z$

and

$$\text{Re} \frac{I^{n+1} F(z)}{I^n F(z)} = \text{Re} \frac{1 - \varepsilon b z}{1 + \varepsilon b z^2} > 0 \left( \varepsilon = (-1)^n, \ b = \frac{c + 2}{c} \right).$$

Thus, $F \in M_n$. From (3.5) we obtain $f(z) = \frac{1}{z} + 2 \frac{c + 1}{c} + \frac{c + 2}{c} z$, $I^n f(z) = \frac{1}{z} + \varepsilon b z \left( \varepsilon = (-1)^n, \ b = \frac{c + 2}{c} \right)$ and

$$\text{Re} \frac{I^{n+1} f(z)}{I^n f(z)} = \text{Re} \frac{1 - \varepsilon b z^2}{1 + \varepsilon b z^2} > 0 \ (z \in E).$$

Thus, $f \in M_n$. Here $f'(z_0) = 0$ and $\text{Re} \frac{I^{n+1} f(z_0)}{I^n f(z_0)} = 0$. Also, the result is sharp for the function $F(z) = \frac{1}{z} + 2 + z$.
REFERENCES


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ÖZET

Bu makalede üstelenmiş Hadamard çarpımı yardımıyla tanımlanan sınıfların yalmıkatiği gösterilmektedir. Ayrıca fonksiyonların integral dönüşümü incelenmektedir.