THE GEOMETRIC INTERPRETATION OF THE SECTIONAL CURVATURE OF A FINSLER SPACE

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Given a generalized Finsler space \( M \) the manifold \( V(M) \rightarrow T(M) \rightarrow 0 \) of all tangent directions on \( M \) admits a naturally induced pseudo-Riemannian structure. Also, there is a linear connection on \( V(M) \) corresponding to the Miron connection \([\theta]\) of \( M \); in terms of the associated exponential formalism on \( V(M) \) the following geometric interpretation of the vertical sectional curvature \( s \) occurs: if \( p \) is a Finslerian 2-plane on \( M \) then \( s(p) \) approximates the difference between the length of a circumference centred at the origin in \( p \) and the length of its exponential projection on \( V(M) \).

1. NOTATIONS, CONVENTIONS AND BASIC FORMULAE

Let \( M \) be an \( n \)-dimensional \( C^\infty \)-differentiable manifold and \( \pi : V(M) \rightarrow M \) the natural projection, where \( V(M) = T(M) - 0 \), while \( T(M) \rightarrow M \) stands for the tangent bundle over \( M \). Let \( \pi^{-1} T(M) \rightarrow V(M) \) be the pullback bundle of \( T(M) \) by \( \pi \). This is a real differentiable vector bundle of rank \( n \).

A generalized metrical Finsler structure on \( M \) is a non-degenerated symmetric Finsler \((0,2)\)-tensor field \( g, g \in \Gamma(V(M), \pi^{-1} T^*(M) \otimes \pi^{-1} T^*(M)) \). Throughout, if \( E \rightarrow N \) is a given vector bundle over the manifold \( N \), then \( \Gamma(N, E) \) denotes the module (over the ring \( C^\infty(N) \) of all real valued smooth functions on \( N \)) of all smooth cross-sections in \( E \). A pair \((M, g)\) is a generalized Finsler space, cf. R.MIRON, [9]. A non-linear connection on \( V(M) \) is a differential system \( N : u \rightarrow T_u(V(M)) \) on \( V(M) \) such that:

\[
T_u(V(M)) = N_u \oplus \ker (d_u \pi)
\]  

for each tangent direction \( u \in V(M) \) on \( M \). See W.BARTHEL, [1]. Consequently \((V(M), N)\) turns to be a non-holonomic space, in the sense of G.VRANCEANU, [19].

Next we consider the bundle epimorphism \( L \) given by \( L : T(V(M)) \rightarrow \pi^{-1} T(M) \), \( L_u \tilde{X} = (u, (d_u \pi) \tilde{X}) \), for any \( u \in V(M) \), \( \tilde{X} \in T_u(V(M)) \). Note that \( \ker(L) = \ker(d \pi) \); thus, if some non-linear connection \( N \) on \( V(M) \) is fixed, each \( L_u : N_u \rightarrow \pi_u^{-1} T(M) \) is a \( IR \)-linear isomorphism, where \( \pi_u^{-1} T(M) = \{u\} \times T_{n(u)}(M) \) denotes the fibre over \( u \) in \( \pi^{-1} T(M) \). We set \( \beta_u = (L|_{N_u})^{-1}, \ u \in V(M) \). The resulting bundle
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isomorphism \( \beta : \pi^{-1} T(M) \to N \) is referred to as the horizontal lift associated with \( N \).

Let \((U, x')\) be a local coordinate system on \( M \) and let \((\pi^{-1}(U), x', y^i)\) be the induced local coordinates on \( V(M) \). Locally, cf. [1], a non-linear connection \( N \) on \( V(M) \) is given by a Pfaffian system:

\[
\delta y^i \equiv dy^i + N^i_j(x, y) \, dx^j = 0. \tag{1.2}
\]

To state this in modern language, let \( X_i : \pi^{-1}(U) \to \pi^{-1} T(M), \ X_i(u) = \left( u, \frac{\partial}{\partial x^i} \right) \right|_{\pi(u)}, \) for any \( u \in \pi^{-1}(U) \). Next, let us set \( \delta_i = \beta X_i, 1 \leq i \leq n \). Let us put

\[
\partial_i = \frac{\partial}{\partial x^i}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i} \quad \text{for simplicity. Then there exists a uniquely determined system of } n^2 \text{ smooth functions } N^i_j \in C^\infty(\pi^{-1}(U)) \text{ such that } \delta_i = \partial_i - N^i_j \dot{\partial}_j \text{ and } N^i_j \text{ are usually termed the coefficients of the non-linear connection } N \text{ with respect to } (U, x'). \text{ Now (1.2) means that, for any } u \in \pi^{-1}(U), N_u \text{ is spanned by } \{ \delta_i \}_1 \leq i \leq n \text{ over the reals.}
\]

The vertical lift is the bundle isomorphism \( \gamma \) defined by \( \gamma : \pi^{-1} T(M) \to \text{Ker}(d\pi), \gamma(X_i) = \dot{\partial}_i \). The definition of \( \gamma \) does not depend upon the choice of local coordinates.

Let \( P_{1,u}, P_{2,u} \) be the natural projections associated with the direct sum decomposition (1.1). We shall need the bundle morphisms:

\[
P_3 = \gamma \circ L, \quad P_4 = \beta \circ G \tag{1.3}
\]

where \( G : T(V(M)) \to \pi^{-1} T(M) \) denotes the Dombrowski mapping, i.e. \( G_u \hat{X} = \gamma_u^{-1} \bar{X}_u \), where \( \bar{X}_u = P_{2,u} \hat{X}, \hat{X} \in T_u(V(M)), u \in V(M) \). Cf. P. DOMBROWSKI, [1].

Let \((M, g)\) be a generalized Finsler space. Each fibre \( \pi_u^{-1}(T(M), u \in V(M)) \) of the pullback bundle carries a semi-definite inner product \( g_u \) and \( u \to g \) is smooth. Therefore \( \pi^{-1} T(M) \to V(M) \) turns into a pseudo-Riemannian vector bundle. Moreover \( V(M) \) admits the pseudo-Riemannian metric:

\[
\tilde{g}(\tilde{X}, \tilde{Y}) = g(L \tilde{X}, L \tilde{Y}) + g(G \tilde{X}, G \tilde{Y}) \tag{1.4}
\]

for any \( \tilde{X}, \tilde{Y} \in \Gamma(V(M), T(V(M))) \) and some fixed non-linear connection \( N \) on \( V(M) \) (with respect to which the Dombrowski map \( G \) is derived). If \( g \) is positive-definite then \( (V(M), \tilde{g}) \) turns to be a \( 2n \)-dimensional smooth Riemannian manifold.

Let \( \nabla \) be a connection in the pullback bundle \( \pi^{-1} T(M) \) of a given generalized Finsler space \((M, g)\). In contrast with the general situation of a connection in an
arbitrary vector bundle, given a non-linear connection $N$ on $V(M)$, two concepts of torsion might be associated with $\nabla$:

$$\tilde{T} (\tilde{X}, \tilde{Y}) = \nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{Y}} \tilde{X} - \tilde{L}[\tilde{X}, \tilde{Y}]$$

$$\tilde{T}_1 (\tilde{X}, \tilde{Y}) = \nabla_{\tilde{X}} G \tilde{Y} - \nabla_{\tilde{Y}} G \tilde{X} - G[\tilde{X}, \tilde{Y}]$$

(1.5)

for any tangent vector fields $\tilde{X}, \tilde{Y}$ on $V(M)$. Nevertheless, note that only the definition of $\tilde{T}_1$ depends on the choice of $N$. Next we consider:

$$T(X, Y) = \tilde{T}(\beta X, \beta Y), \quad S^1(X, Y) = \tilde{T}_1(\gamma X, \gamma Y)$$

(1.6)

for any $X, Y \in \Gamma(V(M), \pi^{-1} T(M))$. We shall need the following result, cf. [7]:

**Theorem 1.1.** There exists a unique connection $\nabla$ in the pullback bundle $\pi^{-1} T(M)$ of the generalized Finsler space $(M, g, N)$ such that the following axioms are satisfied:

$$\nabla g = 0$$

(1.7)

$$T = 0, \quad S^1 = 0.$$  

(1.8)

Moreover $\nabla$ is expressed by:

$$2g(\nabla_\beta X Y, Z) = g(Z, L[\beta X, \beta Y]) - g(X, L[\beta Y, \beta Z]) -$$

$$- g(Y, [\beta X, \beta Z]) - (\beta X)(g(Y, Z)) -$$

$$- (\beta Y)(g(Z, X)) + (\beta Z)(g(X, Y))$$

(1.9)

$$2g(\nabla_\gamma X Y, Z) = g(Z, G[\gamma X, \gamma Y]) - g(X, G[\gamma Y, \gamma Z]) -$$

$$- g(Y, G[\gamma X, \gamma Z]) - (\gamma X)(g(Y, Z)) -$$

$$- (\gamma Y)(g(Z, X)) + (\gamma Z)(g(X, Y))$$

(1.10)

for any $X, Y, Z \in \Gamma(V(M), \pi^{-1} T(M))$.

Next we consider the linear connection $\tilde{\nabla}$ on $V(M)$ defined by:

$$\tilde{\nabla}_X \tilde{Y} = \beta \nabla_{\beta X} L \tilde{Y} + \gamma \nabla_{\gamma X} G \tilde{Y}$$

(1.11)

where $\nabla$ is the connection in $\pi^{-1} T(M)$ furnished by Theorem 1.1. The following result holds:

**Theorem 1.2.** Let $(M, g)$ be a generalized Finsler space carrying the non-linear connection $N$. Then the linear connection (1.11) is subject to:

$$\tilde{\nabla} \tilde{g} = 0.$$  

(1.12)

$$\tilde{\nabla} P_j = 0, \quad j \in \{1, 2, 3, 4\}.$$  

(1.13)

If $\tilde{A}$ is the torsion 2-form of $\tilde{\nabla}$ then:
\[ \tilde{A}(\tilde{X}, \tilde{Y}) = \beta \tilde{T}(\tilde{X}, \tilde{Y}) + \gamma \tilde{T}_1(\tilde{X}, \tilde{Y}) \] (2.14)

for any tangent vector fields \( \tilde{X}, \tilde{Y} \) on \( V(M) \).

The proof of Theorem 1.2, being straightforward, is left as an exercise to the reader.

2. EXPONENTIAL FORMALISM ON A GENERALIZED FINSLER SPACE

Let \((M, g)\) be a generalized Finsler space carrying the non-linear connection \( N \). Consider the linear connection (1.11) on the pseudo-Riemannian manifold \((V(M), \tilde{g})\). Let \( u_0 \in V(M) \) be a fixed tangent direction on \( M \). Let:

\[ \exp_{u_0} : W_\tilde{0} \rightarrow W_{u_0} \] (2.1)

be the exponential mapping associated with the linear connection (1.11), where \( W_\tilde{0} \) and \( W_{u_0} \) are suitable chosen open neighborhoods of the zero tangent vector \( \tilde{0} \) in \( T_{u_0}(V(M)) \), and of \( u_0 \) in \( V(M) \), respectively. On the other hand, for any Finsler space \( M \), there is an exponential formalism associated with the Cartan connection of \( M \), such as developed in B.T.HASSAN, [7]. This might be related to (2.1) as follows: Let \( E : T(M) \rightarrow [0, +\infty) \) be a fixed Finsler energy on \( M \). If the generalized Finsler metric \( g \) is positive-definite and its (local) components are subject to \( g_{ij} = \frac{1}{2} \partial_i \partial_j E \), then \((M, g)\) is a Finsler space. Moreover suppose that \( N \) is (locally) given by:

\[ N_j^i = \frac{1}{2} \partial_j \gamma_{00}^i \] (2.2)

where:

\[ \gamma_{00}^i = \dot{y}^i \dot{y}^j \dot{y}^k , \quad \gamma_{jk}^i = g^{ih} jk , h | \]

\[ | jk , h | = \frac{1}{2} ( \partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk} ) . \]

Then the Miron connection (1.9)-(1.10) coincides with the unique regular Cartan connection of \((M, E)\), such as introduced in E.CARTAN, [7].

Let \( x_0 = \pi(u_0), x_0 \in M \). Put next \( L(u) = E(u)^{1/2} \), for any \( u \in V(M) \). We shall use the following, [7]:

Theorem 2.1. Let \((M, E)\) be a Finsler space and \( \nabla \) its Cartan connection. Then there exists \( \varepsilon > 0 \) such that the following second order ordinary differential system:
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\[ \frac{\partial C}{\partial t} L \frac{dC}{dt} = 0 \] (2.3)

admits a unique solution \( C = C_{x_0}, C_{x_0} : (-2, 2) \rightarrow M \) satisfying the initial conditions \( C_{X_0}(0) = x_0, \) and \( \frac{dC_{X_0}}{dt}(0) = X_0, X_0 \in T_{X_0}(M), \) provided that \( L(X_0) < \varepsilon. \)

To make the notation in (2.3) clear, we mention that given a regular curve \( C : I \rightarrow M, \) for some open interval \( I \subset IR, \) one denotes by \( \bar{C} : I \rightarrow V(M) \) the natural lift of \( C, \) i.e. \( \bar{C}(t) = \frac{dC}{dt}(t), t \in I. \) We shall need the following:

**Theorem 2.2.** The natural lift \( \bar{C} \) of any solution \( C \) of (2.3), i.e. of any geodesic of the Finsler space \((M, E), \) is a horizontal auto-parallel curve of the linear connection (1.11). That is:

\[ \bar{\nabla} \frac{d\bar{C}}{dt} = 0 \] (2.4)

\[ \frac{d\bar{C}}{dt}(t) \in N_{\bar{C}(t)} \] (2.5)

for any value of the parameter \( t. \)

See [4]. There is \( \delta > 0 \) such that the open set:

\[ \{ \bar{X} \in T_{u_0}(V(M)) \mid \tilde{g}_{u_0} (\bar{X}, \bar{X})^{1/2} < \delta \} \]

is contained in \( W_0. \) If \( u_0 \in V(M) \) is chosen such that \( L(u_0) < \varepsilon, \) then according to Theorem 2.1., there is a unique solution \( u_{a_0} : (-2, 2) \rightarrow M \) of (2.3) with initial data \((x_0, u_0).\) We may put:

\[ \exp_{x_0} u_0 = C_{u_0}(1). \] (2.6)

By our Theorem 2.2, the natural lift \( \bar{C}_{u_0} \) of \( C_{u_0} \) is a solution of (2.4). Note also that \( \bar{C}_{u_0}(0) = u_0. \) Next we set

\[ \bar{X}_0 = \frac{d\bar{C}_{u_0}}{dt}(0), \bar{X}_0 \in T_{u_0}(V(M)). \]

Let \( p = \min(\varepsilon, \delta) > 0. \) We establish firstly the following:

**Lemma 2.1.** If \( L(u_0) < p \) then \( \bar{X}_0 \in W_0. \)

**Proof.** It is enough to prove that \( \tilde{g}_{u_0} (\bar{X}_0, \bar{X}_0)^{1/2} < p. \) Let \( v \) be the Liouville vector field on \( M, \) i.e. \( v \in \Gamma(V(M)), \pi^{-1} T(M), v(u) = (u, u), u \in V(M). \) We use now the property (2.5) of \( \bar{C}_{u_0} \) and the definition (1.4). By the classical Euler theorem on positively homogeneous functions one has:
\[ \tilde{g}_{u_0}(\tilde{X}_0, \tilde{X}_0) = g_{u_0} \left( \frac{d C_{u_0}}{dt}(0), \frac{d C_{u_0}}{dt}(0) \right) = g_{u_0}(v(0), v(0)) = E(u_0) \]

and the proof is complete.

By our Lemma 2.1., if \( L(u_0) < \rho \) then:
\[ \exp_{u_0} \tilde{X}_0 = \tilde{C}_{u_0}(1). \] (2.7)

Therefore, the link between the exponentials (2.6) - (2.7) is expressed by:
\[ \pi(\exp_{u_0} \tilde{X}_0) = \exp_{u_0} u_0. \] (2.8)

3. SECTIONAL CURVATURE OF GENERALIZED FINSLER SPACES

Let \((M, g)\) be a generalized Finsler space. Suppose from now on that \( g \) is positive-definite. The 2-dimensional linear subspaces of the fibres of the pullback bundle \( \pi^{-1}T(M) \) give rise to a bundle \( GF_2(M) \) over \( V(M) \), with projection \( p: GF_2(M) \to V(M) \) and standard fibre the Grassman manifold \( G_{2,n} \) of all 2-planes in \( IR^n \). The synthetic object \( GF_2(M) \) over \( V(M) \) is called the Finsler-Grassmann bundle of \( M \). Let \( u_0 \in V(M) \) be a fixed tangent direction on \( M \) and \( p \in GF_2(M), p(p) = u_0 \). Let \( N \) be a non-linear connection on \( V(M) \) and \( \beta \) the corresponding horizontal lift. Let \( \tilde{p} : G_2(V(M)) \to V(M) \) be the Grassmann bundle of all 2-planes tangent to \( V(M) \). We set \( \gamma(p) = \{ \gamma X | X \in p \} \) and \( \beta(p) = \{ \beta X | X \in p \} \). Then \( \gamma(p), \beta(p) \in G_2(V(M)) \). Moreover, if \( \{ X, Y \} \) is an orthonormal basis of \( p \) (with respect to \( g_{u_0} \)) then \( \{ \gamma X, \gamma Y \} \) are basis in \( \gamma(p) \), \( \beta X, \beta Y \) respectively (orthonormal with respect to the inner product \( g_{u_0} \)). Let \( \tilde{B} \) be the curvature 2-form of the linear connection (1.11). As a consequence of (1.12) the (0,4)-tensor field \( \tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \tilde{g}(\tilde{B}(\tilde{Z}, \tilde{W}) \tilde{Y}, \tilde{X}) \) verifies:
\[
\tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) + \tilde{B}(\tilde{X}, \tilde{Y}, \tilde{W}, \tilde{Z}) = 0
\]
\[
\tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) + \tilde{B}(\tilde{Y}, \tilde{X}, \tilde{Z}, \tilde{W}) = 0.
\] (3.1)

Since (3.1) holds, we may consider the (well-defined) map \( b : G_2(V(M)) \to IR \), \( b(p) = \tilde{B}_u(\tilde{X}_0, \tilde{Y}_0, \tilde{X}_0, \tilde{Y}_0), p \in G_2(V(M)) \), for any orthonormal (with respect to \( g_{u_0} \)) linear basis \( \{ \tilde{X}_0, \tilde{Y}_0 \} \) in \( \tilde{p}, u = \tilde{p}(\tilde{p}). \) Next we define \( r, s : GF_2(M) \to IR \), by \( r(p) = b(\beta(p)), s(p) = b(\gamma(p)), p \in GF_2(M) \). The maps \( r, s \) are the horizontal (resp. vertical) sectional curvatures of the Finsler space \((M, E)\), such as
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introduced in [5], provided that \( g \) is given by \( g_{ij} = \frac{1}{2} \partial_i \partial_j E \). Indeed, let \( \tilde{R} \)
be the curvature 2-form of the Miron connection (1.9) - (1.10). Consider the tensor fields \( \tilde{R}(X, Y, \tilde{Z}, \tilde{W}) = g(\tilde{R}(\tilde{Z}, \tilde{W}) Y, X) \), and \( R(X, Y, Z, W) = R(X, Y, \beta Z, \beta W) \), \( S(X, Y, Z, W) = \tilde{R}(X, Y, \gamma Z, \gamma W) \). Then the following identities hold:

\[
\begin{align*}
\tilde{B}(\tilde{X}, \tilde{Y}) \tilde{Z} &= \tilde{\beta} \tilde{R}(\tilde{X}, \tilde{Y}) L \tilde{Z} + \gamma \tilde{R}(\tilde{X}, \tilde{Y}) G \tilde{Z} \\
\tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= \tilde{R}(L \tilde{X}, L \tilde{Y}, \tilde{Z}, \tilde{W}) + \tilde{R}(G \tilde{X}, G \tilde{Y}, \tilde{Z}, \tilde{W})
\end{align*}
\]

and consequently \( r(p) = R(X, Y, X, Y) \), \( s(p) = S(X, Y, X, Y) \), for \( p \in GF_{2}(M) \)
and for any orthonormal linear basis \( \{X, Y\} \) in \( p \).

4. MAIN RESULT

Let \( p_0 \in GF_{2}(M) \), \( u_0 = p(p_0) \), be fixed. Let \( \{X, Y\} \) be an orthonormal basis
in \( p_0 \). Consider the curve \( \eta : [0, 2\pi] \to p_0 \) defined by \( \eta(0) = (\cos \theta) X + \\
(\sin \theta) Y, 0 \leq \theta \leq 2\pi \). For simplicity we set \( p_0^h = \beta(p_0) \), \( p_0^\gamma = \gamma(p_0) \); therefore \( \theta \mapsto \beta(\eta(\theta)) \) (resp. \( \theta \mapsto \gamma(\eta(\theta)) \)) is a curve in \( p_0^h \) (resp. in \( p_0^\gamma \)). With
standard arguments) there exists a number \( r > 0 \) such that:

\[
\begin{align*}
t \beta \eta(\theta) &\in W_0 \cap N_{u_0} \\
t \gamma \eta(\theta) &\in W_0 \cap \text{Ker}(d_{u_0} \pi)
\end{align*}
\]

for any \( 0 \leq t \leq r \). Therefore, the following curves are well defined, i.e.
\( C_0^h, C_0^\gamma : [0, r] \to V(M) \) given by:

\[
C_0^h(t) = \exp_{u_0} t \beta \eta(\theta), \quad C_0^\gamma(t) = \exp_{u_0} t \gamma \eta(\theta)
\]

for any \( 0 \leq \theta \leq 2\pi \), \( 0 \leq t \leq r \). Moreover we consider the curves \( C^h, C^\gamma : [0, 2\pi] \to V(M) \) given by:

\[
C^h(\theta) = C_0^h(\theta), \quad C^\gamma(\theta) = C_0^\gamma(\theta).
\]

Let \( L(C^h), L(C^\gamma) \) be respectively given by

\[
L(C^h) = \int_{0}^{2\pi} \tilde{g}_{C(\theta)} \left( \frac{d C^h}{d \theta} (\theta), \frac{d C^h}{d \theta} (\theta) \right) d \theta,
\]

\[
L(C^\gamma) = \int_{0}^{2\pi} \tilde{g}_{C(\theta)} \left( \frac{d C^\gamma}{d \theta} (\theta), \frac{d C^\gamma}{d \theta} (\theta) \right) d \theta.
\]

We may formulate the following:
Theorem 4.1. Let \((M, g)\) be a generalized Finsler space carrying the nonlinear connection \(N\). Let \(s : GF_2(M) \rightarrow IR\) be the vertical sectional curvature associated with the Miron connection determined by the pair \((g, N)\). Then:

\[
s(p_0) = \lim_{r \to 0} \frac{3}{\pi r^3} \{ I(C^v) - 2 \pi r \}
\]

(4.4)

for each \(p_0 \in GF_2(M)\), where \(C^v\) is given by (4.2).

It is an open problem to establish a geometrical interpretation similar to (4.4) for the horizontal sectional curvature \(r\) of \((M, g, N)\).

5. JACOBI FIELDS ON GENERALIZED FINSLER SPACES

Let us put \(\alpha^r(0, t) = C^r_0(t)\), \(0 \leq 0 \leq 2\pi\), \(0 \leq t \leq r\), with the notations in §4. By (4.2) it follows that the family \(\{C^r_0\}_{0 \leq 0 \leq 2\pi}\) consists of autoparallel curves of \(\tilde{V}\) with the initial data \((u_0, \gamma W(0))\). Clearly \(\alpha^r\) is a variation of \(C^r_0\), in the sense of \([6, p.63]\), vol.II. Let then \(J^r\) be the infinitesimal variation induced by the variation \(\alpha^r\). We need to recall that \(J^r\) is a vector field along the 2-parameter surface \(\alpha^r\) in \(V(M)\) given by:

\[
J^r(u^r(\theta, t)) = J^r_0(t)
J^r_0(t) = \frac{\partial \alpha^r}{\partial \theta}(\theta, t)

\frac{\partial \alpha^r}{\partial \theta}(\theta, t) = (d_0 a_0) \frac{d}{d\theta} \bigg|_0

\alpha^r_t(\theta) = \alpha^r(\theta, t).
\]

Note that:

\[
J^r_0(0) = 0 , \quad 0 \leq 0 \leq 2\pi.
\]

(5.2)

Let \(u_0 \in V(M)\) be fixed. Put for brevity \(W^r_0 = W_0 \cap \text{Ker}(d_u \pi)\). Consider \(\tilde{X}_0 \in W^r_0\) and the curve \(\gamma_0\) in \(V(M)\) defined by:

\[
\gamma_0(t) = \exp_{u_0} t \tilde{X}_0
\]

(5.3)

for small values of the parameter \(t\). Next we consider the first order ordinary differential system:

\[
\tilde{V}_{d_{\alpha}} \tilde{Z} = 0
\]

(5.4)
where \( \sigma : [0, 1] \to V(M) \) is a given differentiable curve in \( V(M) \). Let then \( T_{\sigma,t}^* : T_{\sigma(0)}(V(M)) \to T_{\sigma(t)}(V(M)) \) be the parallel displacement operator along \( \sigma \), associated with (5.4). That is, if \( \tilde{Z} \) is the unique solution of (5.4) with initial data \( \tilde{Z}(0) = \tilde{Z}_0 \) then \( T_{\sigma,t}^*(\tilde{Z}_0) = \tilde{Z}(t) \), for any \( \tilde{Z}_0 \in T_{\sigma(0)}(V(M)) \). We establish:

**Lemma 5.1.** For an arbitrary smooth curve \( \sigma : [0, 1] \to V(M) \) one has:

\[
P_2 \circ T_{\sigma,t} = T_{\sigma,t}^* \circ P_2
\]

for any \( 0 \leq t \leq 1 \).

**Proof.** Let \( \vec{X} \in \mathcal{T}_\sigma(V(M)) \) and \( \tilde{Z} \) the unique solution of (5.4) with \( \tilde{Z}(0) = P_2 \vec{X} \). Then \( 0 = P_2 \frac{d}{dt} \tilde{Z} = \frac{d}{dt} P_2 \tilde{Z} \), by our (1.13), i.e. \( P_2 \tilde{Z} \) is a solution of (5.4). Moreover \( (P_2 \tilde{Z})'(0) = P_2 P_2 \vec{X} = \tilde{Z}(0) \). Consequently \( P_2 \tilde{Z} = \tilde{Z} \), and (5.5) holds, Q.E.D.

Let us replace now \( \sigma \) in (5.4) by the curve (5.3). By the very definition of \( \gamma_0 \), its tangent gives a solution of (5.4) \( \left( \text{since } \gamma_0 \text{ is an auto-parallel curve of the linear connection (1.11)} \right) \) and \( \frac{d}{dt} \gamma_0(0) = \vec{X}_0 \). Applying Lemma 5.1. one has:

\[
\frac{d}{dt} \gamma_0(t) = T_{\gamma_0,t}^*(\vec{X}_0) = T_{\gamma_0,t}^*(P_2 \vec{X}_0) = P_2 T_{\gamma_0,t}^*(\vec{X}_0) = P_2 \frac{d}{dt} \gamma_0(t).
\]

It follows that (5.3) is a vertical curve provided that \( \vec{X}_0 \) is vertical. Thus:

\[
(d_{\gamma_0(0)} \pi) \frac{d}{dt} \gamma_0(t) = 0
\]

or \( \pi \circ \gamma_0 = \text{constant} \), i.e. the curve (5.3) lies entirely in the fibre \( V_{x_0} = \pi^{-1}(x_0) \subset V(M) \), \( x_0 = \pi(u_0) \). The result obtained in terms of the curve (5.3) might be equally applied to the curve \( C_0^\alpha(t) \) given by (4.2). Therefore:

\[
C_0^\alpha(t) \in V_{x_0}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq t \leq r.
\]

In addition to (5.1) we consider:

\[
\frac{\partial}{\partial t} \alpha^\gamma(\theta, t) = (d_t \alpha^\gamma(\theta, 0)) \frac{d}{dt} t
\]

\[
\alpha^\gamma(t) = \alpha^\gamma(\theta, t).
\]

By (1.14) one has:

\[
\vec{A}(\gamma X, \gamma Y) = \gamma S^1(X, Y) = 0
\]

for any \( X, Y \in \Gamma(V(M), \pi^{-1} T(M)) \). Let us define:
\[
\frac{D}{dt}(\tilde{X}, t) = (\tilde{\nabla}_{\tilde{\omega}_t} \tilde{X})_{u(t, t)} \tag{5.7}
\]

for any tangent vector field \( \tilde{X} \) of \( V(M) \) defined along the 2-parameter surface \( \alpha^\tau \) in \( V(M) \). Since \( C^\alpha_0 \) lies entirely in \( V_{\gamma_0} \), and \( V_{\gamma_0} \) is the maximal integral manifold of the vertical distribution \( \text{Ker} (d\pi) \) passing through \( u_0 \), one obtains:

\[
\frac{\partial \alpha^\tau}{\partial t}(t, t), \quad \frac{\partial \alpha^\tau}{\partial \theta}(0, t) \in \text{Ker} (d_{\alpha(0, t)} \pi) \tag{5.8}
\]

for \( 0 \leq \theta \leq 2\pi, 0 \leq t \leq r \). Using (5.6) - (5.8) we derive:

\[
\frac{D J^\alpha}{dt}(0, 0) = \left\{ \tilde{\nabla}_{\tilde{\omega}_0} \frac{\partial \alpha^\tau}{\partial t} \right\}_{u_0} \tag{5.9}
\]

since \( \left[ \frac{\partial \alpha^\tau}{\partial t}, \frac{\partial \alpha^\tau}{\partial \theta} \right] = 0 \).

6. PROOF OF THE MAIN RESULT

Let \( \tilde{\pi}: T(V(M)) \to V(M) \) be the natural projection of the tangent bundle over \( V(M) \). We consider the natural imbedding \( \eta_t: T(V(M)) \to T(T(V(M))), t \in IR, \) defined as follows: Let \( \tilde{X}_0 \in T(V(M)) \). Consider the curve \( a(t) = t \tilde{X}_0 \) in \( T(V(M)) \). Set:

\[
\eta_t(\tilde{X}_0) = \frac{da}{dt}(t). \tag{6.1}
\]

Actually, if \( \tilde{\pi}(\tilde{X}_0) = u_0, u_0 \in V(M) \), then \( a(t) \) is a curve in \( T_{u_0}(V(M)) \). Therefore, its tangent vector at \( a(t) \) is an element of \( T_{\tilde{X}_0}(T_{u_0}(V(M))) = \text{Ker} (d_{\tilde{X}_0} \tilde{\pi}), t \in IR \). Let us consider now the curve (5.3) with \( \tilde{X}_0 \in W_0 \) not necessarily vertical. We may rewrite it:

\[
\gamma_0(t) = \exp_{u_0} a(t) \tag{6.2}
\]

for small enough values of \( t \); taking the differential of (6.2) at \( t \) furnishes:

\[
\frac{d\gamma_0}{dt}(t) = (d_{a(t)} \exp_{u_0}) \eta_t(\tilde{X}_0). \tag{6.3}
\]

Take (6.3) at \( t = 0 \); since \( \gamma_0 \) is an auto-parallel curve of (1.11) with initial data \( (u_0, \tilde{X}_0) \) it follows:

\[
(d_{u_0} \exp_{u_0}) \eta_0 \tilde{X}_0 = \tilde{X}_0. \tag{6.4}
\]

We apply the results given by (6.3) - (6.4) to the curve \( C^\alpha_0 \). Thus one has:
\[ \frac{\partial \alpha^r}{\partial t} (0, 0) = \gamma W(\theta), \quad 0 \leq \theta \leq 2\pi. \] (6.5)

Let \((x^a) = (x^1, y^i), 1 \leq a \leq 2n,\) be the natural local coordinates on \(V(M).\) Let \(T_{bc}\) be the corresponding local coefficients of the linear connection (1.11). The right hand side of (5.9) is locally given by:

\[ \left\{ \begin{array}{c}
\alpha^r \\
\frac{\partial \alpha^r}{\partial \theta}
\end{array} \right\}_{u_0} = \frac{\partial^2 \alpha^a}{\partial \theta \partial t} (0, 0) + \Gamma_{bc}^a (\alpha^r (0, 0)) \frac{\partial \alpha^b}{\partial \theta} (0, 0) \frac{\partial \alpha^c}{\partial t} (0, 0) \] (6.6)

where \(\alpha^a (0, t) = (\alpha^1 (0, t), ..., \alpha^{2n} (0, t)).\) Let \(W^i (\theta) = x^i \cos \theta + y^i \sin \theta\) be the components of the Finslerian vector field \(W(\theta)\) on \(M.\) Our (6.5) leads to:

\[ \frac{\partial \alpha^i}{\partial t} (0, 0) = 0, \quad \frac{\partial \alpha^{a+i}}{\partial t} (0, 0) = W^i (\theta) \] (6.7)

for \(1 \leq i \leq n.\) By (5.1) - (5.2) and (6.6) - (6.7) one has

\[ \frac{D J^r}{\partial t} (0, 0) = \frac{d W^i}{d \theta} (0) \hat{\theta}_1 \bigg|_{u_0} \]

or:

\[ \frac{D J^r}{\partial t} (0, 0) = \gamma W \left( \theta + \frac{\pi}{2} \right). \] (6.8)

For each \(\vec{X} \in T_u (V(M))\) we put \(\| \vec{X} \| = \langle g_u (\vec{X}, \vec{X}) \rangle^{1/2}.\) We consider the function \(f^r_0 : [0, r] \to (0, +\infty)\) given by:

\[ f^r_0 (t) = \| J^r_0 (t) \|^2, \quad 0 \leq t \leq r. \] (6.9)

We develop (6.9) as a Taylor series:

\[ f^r_0 (t) = \sum_{k=0}^{4} \frac{t^k}{k!} (D^k f^r_0) (0) + o (t^4) \] (6.10)

and compute \(D^k f^r_0\), where \(D^k = \frac{\partial^k}{\partial t^k}, 0 \leq k \leq 4.\) By (5.2), (6.8) one obtains:

\[ f^r_0 (0) = 0 \]

\[ (D f^r_0) (0) = 0 \]

\[ (D^2 f^r_0) (0) = 0 \] (6.11)

since the connection (1.11) verifies (1.12). How (5.1) is the infinitesimal variation induced by the variation \(\alpha^r;\) by Theorem 1.2. in \([5, p.64]\) one obtains:
\[ \tilde{\nabla}^2_{\alpha \gamma} J^\nu + \tilde{\nabla}^2_{\alpha \gamma} \tilde{A} \left( J^\nu, \frac{\partial \alpha^\nu}{\partial t} \right) + \tilde{B} \left( J^\nu, \frac{\partial \alpha^\nu}{\partial t} \right) \frac{\partial \alpha^\nu}{\partial t} = 0. \tag{6.12} \]

Take (6.12) at \( u_0 \). By (5.1), (5.6), (5.8) it turns into:
\[ \{ \tilde{\nabla}^2_{\alpha \gamma} J^\nu \}_u = 0. \tag{6.13} \]

Consequently:
\[ (D^3 f^\nu_0)(0) = 0. \tag{6.14} \]

Let \( S(X, Y) Z = \tilde{R}(\gamma X, \gamma Y) Z \) be the vertical curvature of the Miron connection, \( X, Y, Z \in \Gamma(V(M), \pi^{-1} T(M)) \). By (3.2) one obtains \( B(\gamma X, \gamma Y) \gamma Z = \gamma S(X, Y) Z \).

Using (1.12) we have:
\[ (D^4 f^\nu_0)(0) = 8 \tilde{g}_{u_0} \left( \{ \tilde{\nabla}^2_{\alpha \gamma} J^\nu \}_u, \{ \tilde{\nabla}^2_{\alpha \gamma} J^\nu \}_u \right). \tag{6.15} \]

Take the covariant derivative of the Jacobi equation (6.12) in the direction \( \frac{\partial \alpha^\nu}{\partial t} \). Moreover, take the inner product of the resulting equation by \( \{ \tilde{\nabla}_{\alpha \gamma} J^\nu \}_u \).

Then (6.15) becomes:
\[ (D^4 f^\nu_0)(0) = 8 \tilde{g}_{u_0} \left( \tilde{\nabla}^2_{\alpha \gamma} \tilde{B} \left( J^\nu, \frac{\partial \alpha^\nu}{\partial t} \right) \frac{\partial \alpha^\nu}{\partial t}, \tilde{\nabla}^2_{\alpha \gamma} J^\nu \right). \tag{6.16} \]

On the other hand:
\[ \tilde{\nabla}^2_{\alpha \gamma} \tilde{B} \left( J^\nu, \frac{\partial \alpha^\nu}{\partial t} \right) \frac{\partial \alpha^\nu}{\partial t} = \tilde{B} \left( \tilde{\nabla}^2_{\alpha \gamma} J^\nu, \frac{\partial \alpha^\nu}{\partial t} \right) \frac{\partial \alpha^\nu}{\partial t}. \tag{6.17} \]

Now take (6.17) in \( u_0 \) and use (6.8). From the resulting equation let us substitute in (6.16). We obtain:
\[ (D^4 f^\nu_0)(0) = -8 \tilde{g}_{u_0} \left( \tilde{B} \left( \gamma W \left( \theta + \frac{\pi}{2} \right), \gamma W(\theta) \right) \gamma W(0), \gamma W \left( 0 + \frac{\pi}{2} \right) \right). \tag{6.18} \]

Moreover, in terms of the vertical curvature tensor:
\[ (D^4 f^\nu_0)(0) = -8 S_{u_0} \left( W \left( \theta + \frac{\pi}{2} \right), W(\theta), W \left( 0 + \frac{\pi}{2} \right), W(\theta) \right). \tag{6.19} \]

At this point we may substitute in (6.10) from the formulae (6.11), (6.14) and (6.19). This procedure gives:
\[ f^\nu_0(t) = t^2 \left\{ 1 - \frac{t^2}{3} S_{u_0} \left( W \left( \theta + \frac{\pi}{2} \right), W(\theta), W \left( 0 + \frac{\pi}{2} \right), W(\theta) \right) + o(t^2) \right\}. \tag{6.20} \]

As \( (1 - \delta)^{1/2} = 1 - \frac{1}{2} \delta + o(\delta^2) \) we obtain:
\[ L(C^*) = 2 \pi r + \int_0^{2\pi} \left( W(\theta + \frac{\pi}{2}), W(\theta), W\left(\theta + \frac{\pi}{2}\right), W(\theta) \right) d\theta + o(r^3). \] (6.21)

Now \( \left\{ W(\theta), W\left(\theta + \frac{\pi}{2}\right) \right\} \) is an orthonormal basis in \( p_0 \in GF_2(M), u_0 = \rho(p_0) \), and thus (6.21) leads to (4.4), Q.E.D.

**REFERENCES**


**ÖZET**

Bu çalışmada, genelleştirilmiş bir \( M \) Finsler uzayı verildiğine göre, \( M \) üzerindeki bütün teğet doğrultularının \( V(M) = T(M) - 0 \) manifoldum yapısı incelenmektedir.