AFFINE MOTION IN RECURRENT AREAL SPACES OF SUBMETRIC CLASS

S.P. SINGH

ABSTRACT
We consider affine motion in recurrent areal space of submetric class, denoted by $A_{n}^{\text{sub}}$. The necessary and sufficient conditions are obtained when affine motions admit contra-fields. Other special cases are also discussed.

In a set of papers K. Takano ([1], [2]) has studied affine motion in non-Riemannian $K^{*}$-spaces. Affine motions in recurrent Finsler were studied by R.S. Sinha [5]. T. Igarashi [4] has introduced the theory of Lie-derivative in an areal space of submetric class. The concept of deformed areal spaces and the homothetic transformations in areal space of submetric class was developed by O.P. Singh ([6], [7]). Recently the present author [8] discussed the motion with contra field in a symmetric areal space of submetric class. In this paper the author wishes to study affine motion in recurrent areal space of submetric class.

1 INTRODUCTION

Let us consider an $n$-dimensional areal space $A_{n}^{\text{sub}}$ of submetric class equipped with the fundamental function $F(x^i, p_i^\alpha)$; $p_i^\alpha = \frac{\partial x^i}{\partial u^\alpha}$, the normalized metric tensor $g_{ij}(x,p)$ and symmetric connection parameter $\Gamma^i_{jk}(x,p)$ [3]. Throughout this paper the indices $i, j, k, l$... run from 1 to $n$ while the indices $\alpha, \beta, \lambda, \mu$... vary from 1 to $m$, ($1 \leq m \leq (n-1)$). An areal space $A_{n}^{\text{sub}}$, in which the curvature tensor $R^i_{jkl}$ satisfies the relation

\begin{equation}
R^i_{jkl} = K_m R^i_{jkl},
\end{equation}
where the suffix after the bar denotes covariant differentiation and $K_m$ a non-zero covariant vector, is called recurrent areal space of submetric class. We denote such space by $A_{n}^{*(m)}$.

In an areal space of submetric class $A_{n}^{*(m)}$, we consider an infinitesimal point transformation

$$
\tilde{x}' = x' + \xi'(x)\partial \tau,
$$

where $\xi'(x)$ is a contravariant vector field of class $C^2$ and is only a point function and $\partial \tau$ is a constant.

The Lie-derivative of a mixed tensor $T^i_j$ with respect to (1.2) is given by [4]

$$
L_\xi T^i_j = T^i_j \xi^h - T^i_j \xi^h + T^i_j \xi^h + T^i_j \xi^h + T^i_j \xi^h + T^i_j \xi^h .
$$

where the symbol $L_\xi$ denotes the operator of Lie-differentiation.

Some important formulae concerning the operators of Lie-differentiation and covariant differentiation are expressed as follows:

$$
L_\xi \left( T^i_j \right)_{jk} = T^i_j \xi^h - T^i_j \xi^h - T^i_j \xi^h - T^i_j \xi^h - T^i_j \xi^h - T^i_j \xi^h .
$$

(1.5)

$$
L_\xi \left( T^i_j \right)_{jr} = 0
$$

and

(1.6)

$$
\left( L_\xi \Gamma^i_{jk} \right)_{jl} = L_\xi \Gamma^i_{jk} + \Gamma^i_{jk} \xi^h \Gamma^h_{sl} - \Gamma^i_{jk} \xi^h \Gamma^h_{sl} .
$$

2. AFFINE MOTION IN RECURRENT AREAL SPACE $A_{n}^{*(m)}$

In an areal space of submetric class if the original space and the deformed space with connection parameter $\Gamma^i_{jk} + L_\xi \Gamma^i_{jk} \partial \tau$ have the same connection,
the transformation (1.2) is called affine motion of the space $A_n^{(m)}$. In such case, it is necessary and sufficient that we have

\begin{equation}
L_\xi \Gamma^i_{jk} = 0 .
\end{equation}

Under an affine motion in view of (1.6) and (2.1), we necessarily obtain

\begin{equation}
L_\xi R^i_{jkl} = 0 .
\end{equation}

Applying (1.4) for the curvature tensor $R^i_{jkl}$ and using (2.1), we get

\begin{equation}
L_\xi R^i_{jkl|m} = 0 .
\end{equation}

In view of (2.2) and (2.3), the equation (1.1) yields

\begin{equation}
L_\xi K_m = 0
\end{equation}

since $A_n^{(m)}$ is a non-flat space.

Thus we state

**Theorem 2.1** If a recurrent areal space of submetric class $A_n^{(m)}$ admits the affine motion (1.2) then the recurrence vector $K_m$ is Lie-invariant.

Now we wish to examine the possibility of the existence of an affine motion of the form

\begin{equation}
\vec{x}' = x' + \xi'(x)\delta t , \quad \xi'(x) = \Phi(x, p)S_j
\end{equation}

in the recurrent areal space of submetric class.

Taking Lie-derivative of the curvature tensor $R^i_{jkl}$, we get
(2.6) \[ L^*_t R^I_{jkl} = R^I_{jkl} R^h - R^h R^I_{jkl} + R^I_{jkl} R^h + R^I_{jkl} R^h + R^I_{jkl} R^h + R^I_{jkl} R^h = 0 \]

by virtue of (2.2).

Introducing latter of (2.5) in (2.6), we obtain

(2.7) \[ R^I_{jkl} R^h + 2 \Phi R^I_{jkl} + \Phi R^I_{jkl} + \Phi R^I_{jkl} = 0 \]

Noting (1.1) in (2.7), it becomes

(2.8) \[ (K_h R^h + 2 \Phi) R^I_{jkl} + \Phi R^I_{jkl} = 0 \]

Let us assume that \( R^I_{jkl} P^h = 0 \), then (2.8) reduces to

(2.9) \[ \Phi = -\frac{1}{2} K_h R^h \]

since the space \( A^\ast_{(m)} \) is non-flat.

Conversely, if the relation (2.9) is true, the equation (2.8) takes the form

(2.10) \[ K_m R^I_{jkl} = 0 \]

Since \( K_m R^I_{jkl} = 0 \), the equation (2.10) yields

(2.11) \[ R^I_{jkl} P^h = 0 \]

Accordingly we state

**Theorem 2.2** If \( A^\ast_{(m)} \) admits an affine motion of the form (2.5), the necessary and sufficient condition for \( \Phi(x, p) \) to be expressed in the form

\[ \Phi = -\frac{1}{2} K_h R^h \]
is that the condition $R_{jkl}^{i} P_{p}^{h} = 0$ holds.

3. FURTHER DISCUSSION

In this section, we shall deal with two special cases of the affine motion in a recurrent areal space of submetric class.

(a) Contra Field  In an areal space of submetric class, if the vector $\xi^{l}(x)$ satisfies the relation

$$\xi_{i}^{j} = 0$$

the vector field $\xi^{l}(x)$ is called a contra field.

In this case we consider a special affine motion of the form

$$\bar{x}^{i} = x^{i} + \xi^{l}(x) \delta t$$

$$\xi_{i}^{j} = 0$$

In view of (3.2), the equation (2.1) yields

$$L_{\xi} \Gamma^{ij}_{jk} = R_{jkh}^{i} \xi^{h} = 0.$$  

Applying the latter of (3.2) and (2.2) in the equation (2.6), we get

$$R_{jkl}^{i} \xi^{h} = 0.$$ 

In view of (1.1), it becomes

$$R_{jkl}^{i} K_{h} \xi^{h} = 0.$$ 

But $A_{n}^{*}(m)$ is a non-flat space, that is, $R_{jkl}^{i} \neq 0$, therefore it is obvious that

$$K_{h} \xi^{h} = 0.$$
From the equations (3.3) and (3.6), we conclude that for $A_{n}^{*(m)}$ to admit an affine motion of the form (3.2), it is necessary that we have

$$K_{h} \xi^{h} = 0, \quad R^{i}_{jkh} \xi^{h} = 0. \quad (3.7)$$

Conversely, if (3.7) is true, then from the identity $R^{i}_{jkh} + R^{i}_{khj} + R^{i}_{hjk} = 0$ and the latter of (3.7), we get

$$R^{i}_{jkh} \xi^{h} = 0. \quad (3.8)$$

But by using (3.8) in the Ricci identity

$$\xi_{|j|k}^{i} - \xi_{|k|j}^{i} = R^{i}_{jkh} \xi^{h}, \quad (3.9)$$

we obtain (3.1) and hence $\xi^{i}(x)$ is contra field in $A_{n}^{*(m)}$.

In such case in view (3.6), the equation (2.6) immediately implies that $L_{v} R^{i}_{jkh} = 0$, which is integrability condition of $L_{\xi} \Gamma^{i}_{jk} = 0$. Thus (3.7) is also a sufficient condition for $A_{n}^{*(m)}$ admitting (3.2).

Hence we state

**Theorem 3.1** When a recurrent areal space of submetric class admits an affine motion in order that the vector $\xi^{i}(x)$ spans a contra field, it is necessary and sufficient that the conditions $K_{h} \xi^{h} = 0$ and $R^{i}_{jkh} \xi^{h} = 0$ be valid.

**(b) Concurrent Field** In an areal space of submetric class, if the vector $\xi^{i}(x)$ satisfies the relation

$$\xi_{|j|}^{i} = K \delta^{i}_{j}, \quad (3.10)$$

where $K$ is a constant, then the vector field $\xi^{i}(x)$ is called a concurrent vector field.
Here we consider the affine motion

\[ x' = x + \xi'(x) \partial t, \quad \xi'_{\ell} = K \delta_{\ell}^j. \] (3.11)

From the latter of (3.11), we find

\[ \xi_{\ell/k} + \xi_{\ell/j} = 0. \] (3.12)

Application of (3.12) in the Ricci identity (3.9) yields \( R^{i}_{\ell/h} \xi^h = 0. \)

Taking covariant differentiation of the above equation and noting (1.1) and (3.10), we obtain

\[ KR^{i}_{\ell/m} = 0. \] (3.13)

Since \( K \) is non zero constant, the equation (3.13) implies

\[ R^{i}_{\ell/m} = 0, \] (3.14)

which contradicts our assumption that the space \( A^{(m)}_n \) is non flat.

Accordingly, we state

**Theorem 3.2** The general recurrent areal space of submetric class \( A^{(m)}_n \) does not admit the affine motion (3.11).

**REFERENCES**


S.P. SINGH
Department of Mathematics,
Egerton University,
P.O. Box 536, Njoro,
Kenya
E-Mail:drgatoto@yahoo.com