2-Transitive Frobenius Q-Groups

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Abstract. A Q-group is a finite group all of whose irreducible complex characters are rationally-valued. In this paper, we find all 2-transitive Frobenius Q-groups.

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INTRODUCTION

We use the following notations in this paper

Notation. C the field of complex numbers; Z the ring of rational integers; \(\text{Gen}(x)\) the set of generators of cyclic group \(<x>\); \([x]\) the conjugacy class of \(x\); \(N_G(x)\) the normalizer of \(x\) in \(G\); \(C_G(x)\) the centralizer of \(x\) in \(G\); \(\text{Aut}(x)\) the group of automorphisms of \(<x>\); \(\varphi\) Euler’s function; \(\times\) semidirect product; \(Q_8\) the quaternion group of order 8; \(E(p^n)\) the elementary abelian \(p\)-group of order \(p^n\).

Q-GROUPS

Definition. A finite group whose complex characters are rationally-valued is called a Q-group.

For example, all of the symmetric groups and finite elementary abelian 2-groups are Q-groups. Kletzing’s lecture notes, [3], presents a detailed investigation into the structure of Q-groups. General classification of Q-groups has not been able to be done up to now, but some special Q-groups have been classified. Frobenius Q-groups were classified by [1].
Let $G$ be a finite group of order $n$ and let $\xi$ be a primitive $n$-th root of unity in the field $C$. Then, all of the complex character values of $G$ lie in the subring $\mathbb{Z}[\xi]$ of $C$. Moreover, if $G$ is a $Q$-group, these values lie in $\mathbb{Z}$ since $\mathbb{Z}[\xi] \cap Q = \mathbb{Z}$. Now, we say that a finite group is a $Q$-group if and only if the values of its all the irreducible complex characters lie in $\mathbb{Z}$.

Another characterization of $Q$-groups is the following theorem.

**Theorem 1.** Let $G$ be a finite group. Then, $G$ is a $Q$-group if and only if for every $x \in G$, $\text{Gen}(x) \subseteq [x]$ i.e. for every $x \in G$, $N_G(x)/C_G(x) \cong \text{Aut}(x)$ [3].

Now, we can easily see the following corollary.

**Corollary 2.** Let $G$ be a $Q$-group. Then:

1) $[N_G(x):C_G(x)] = |\text{Aut}(x)| = \varphi(|x|)$, for every $x \in G$.

2) If $G \neq \{1\}$, $2||G||$.

3) If $N$ is a normal subgroup of $G$, then $G/N$ is a $Q$-group.

**FROBENIUS GROUPS**

**Definition.** Let $G$ be a transitive and non-regular permutation group on $\Omega$, $|\Omega| \in IN$, $\alpha \in \Omega$, $H = G_{\alpha}$. Then, $G$ is called a Frobenius group with complement $H$ if and only if the identity element of $G$ is unique element that fixes more than one element of $\Omega$.

**Definition.** A trivial intersection set in a group $G$ is a subset $S$ of $G$ such that for all $g \in G$, either $S^g = S$ or $S^g \cap S \subseteq \{1\}$.

**Lemma 3.** Let $G$ be a finite group, $\{1\} \neq H$ a proper subgroup of $G$. Then the following are equivalent:
(a) \( G \) is a Frobenius group with complement \( H \).

(b) \( H \) is a trivial intersection set and \( H = N_G(H) \). [2]

Now we can say that a finite group \( G \) is a Frobenius group if and only if it contains a proper subgroup \( H \neq \{1\} \), called a Frobenius complement, such that \( H \cap H^x = \{1\} \) for all \( x \not\in H \).

By Frobenius Theorem [2, p.63], a Frobenius group \( G \) with complement \( H \) has a normal subgroup \( K \), called Frobenius kernel, such that \( HK = G \). If \( K = \{x_1, \ldots, x_n\} \) where \( n \in \mathbb{N} \), then we have \( G = K \cup \bigcup_{i=1}^{n} (H^{x_i} - \{1\}) \), called Frobenius partition.

**Definition.** Let \( G \) be a group and \( \varphi \) be an automorphism of \( G \). Then, \( \varphi \) is called fixed-point-free automorphism if \( \varphi(g) \neq g \) for every \( g \in G - \{1\} \).

Let \( G \) be a Frobenius group with kernel \( K \) and complement \( H \). Then, for every \( h \in H - \{1\} \), \( k \mapsto h^{-1} kh \) is an automorphism of \( K \) fixing only the element \( 1 \in K \). Thus we can say that all elements of \( H \) except 1 are fixed-point-free of \( K \). Moreover, a semi-direct group \( G = K \rtimes H \) is a Frobenius group if \( h \) is fixed-point-free of \( K \) for every \( h \in H - \{1\} \).

**Lemma 4.** Let \( G \) be a transitive permutation group on \( \Omega \), \( |\Omega| \in \mathbb{N} - \{1\} \). Then \( G \) is 2-transitive if and only if \( G = G_\alpha \cup G_\alpha x G_\alpha \) for all \( \alpha \in \Omega \) and \( x \in G - G_\alpha \) [4].

**Theorem 5.** Let \( G \) be a Frobenius group with kernel \( K \) and complement \( H \). Then \( G \) is 2-transitive if and only if \( |K| = |H| + 1 \).

**Proof.** By the definition of Frobenius group, \( G \) is a transitive permutation group on \( \Omega \), \( |\Omega| \in \mathbb{N} \) and there is \( \alpha \in \Omega \) such that \( H = G_\alpha \). By Lemma 4., we know that \( G \) is 2-transitive if and only if \( G = H \cup H x H \) for every \( x \in G - H \). Therefore, if \( G \) is 2-transitive, then we have
\[ |G| = |H| + |H \times H| = |H| + \frac{|H| \cdot |H^2|}{|H \cap H^2|} = |H| + \frac{|H|^2}{|H \cap H^2|} \]

for every \( x \in G - H \). Since \( H \) is a trivial intersection set in \( G \) and \( H = N_G(H) \) by Lemma 3., we have \( |H \cap H^2| = 1 \) for every \( x \in G - H \). Thus, \( |G| = |H| + |H|^2 \). Also, since \( |G| = |K| \cdot |H| \) by Frobenius Theorem, we have \( |K| \cdot |H| = |G| = |H| + |H| \) and so \( |K| = |H| + 1 \). Conversely, we can easily that if \( |K| = |H| + 1 \), then \( G \) is 2-transitive.

**Theorem 6.** Let \( G \) be a Frobenius \( Q \)-group with kernel \( K \) and complement \( H \). Then, \( H \cong Z_2 \) or \( H \cong Q_8 \). Moreover,

1) If \( H \cong Z_2 \), then \( K \) is an elementary abelian 3-group and for every \( t \in K \),

\( t^u = t^{-1} \) where \( 1 \neq u \in H \).

2) If \( H \cong Q_8 \), then \( K \) is an elementary abelian \( p \)-group where \( p = 3 \) or \( p = 5 \). [1]

**2-TRANSITIVE FROBENIUS \( Q \)-GROUPS**

All 2-transitive Frobenius \( Q \)-groups are given by the following theorem.

**Theorem.** Let \( G \) be a 2-transitive Frobenius \( Q \)-group. Then, \( G \cong S_3 \) or \( G \cong E(3^2) \times Q_8 \) where \( E(3^2) \) is the 2-dimensional irreducible module of group algebra \( Z_3 Q_8 \).

**Proof.** Let \( G \) be a 2-transitive Frobenius \( Q \)-group with kernel \( K \) and complement \( H \). By Theorem 6, we know that \( H \cong Z_2 \) or \( H \cong Q_8 \).

1) If \( H \cong Z_2 \), then \( K \) is an elementary abelian 3-group and for every \( t \in K \),

\( t^u = t^{-1} \) where \( 1 \neq u \in H \) by Theorem 6. Moreover, since \( G \) is 2-transitive, we have \( |K| = |H| + 1 = 3 \) by Theorem 5. Therefore, \( G \cong E(3) \times Z_2 \cong S_3 \). Conversely, we can see easily that \( S_3 \) is a 2-transitive Frobenius \( Q \)-group.
2) If \( H \cong Q_8 \), then \( K \) is an elementary abelian \( p \)-group where \( p = 3 \) or \( p = 5 \) by Theorem 6. Since \( G \) is 2-transitive, we have \(|K| = |H| + 1 = 9\) by Theorem 5. Then, \( K \) must be an elementary abelian 3-group of order 9. Since \( K \triangleleft G \), \( H \) acts on \( K \) by conjugation. Thus, \( K \) may be considered as a \( Z_3H \)-module, so \( K \) defines a representation of \( H \) over the field \( Z_3 \). Since \( 3| |H| \), we can use ordinary representation theory, so \( K \) is a direct sum of some irreducible modules of group ring \( Z_3H \) by Maschke’s Theorem and Wedderburn’s Theorem. \( H \) has exactly five non-isomorphic irreducible module over the field \( Z_3 \) and four of them are 1-dimensional so the other is 2-dimensional. Since \( \chi(u) = 1 \in Z_3 \) for every 1-dimensional representation \( \chi \) of \( H \) where \( u \) is the involution, \( K \) must be the 2-dimensional irreducible module of group ring \( Z_3H \). Therefore, we have \( G \cong E(3^2) \times |Q_8| \) where \( E(3^2) \) is the 2-dimensional irreducible module of group algebra \( Z_3Q_8 \). Conversely, let \( G \cong E(3^2) \times |Q_8| \) where \( E(3^2) \) is the 2-dimensional irreducible module of group algebra \( Z_3Q_8 \). Then, \( G \) is a Frobenius group with kernel \( K \) \((\cong E(3^2))\) and complement \( H \) \((\cong Q_8)\) since the involution of \( H \) is fixed-point-free of \( K \). Moreover, since \(|K| = |H| + 1\), \( G \) is 2-transitive by Theorem 5. Since \( G \) is a Frobenius group, we have Frobenius partition \( G = K \cup \left( \bigcup_{g \in K} (H^g - \{1\}) \right) \). Also, for every \( g \in K \), \( H^g \) \((\cong Q_8)\) is a \( Q \)-group and for every \( 1 \neq x \in K \), \( x^u = x^2 \) where \( u \) is the involution of \( H \). Thus, by the definition of \( Q \)-group, we can see easily that \( G \) is a \( Q \)-group.

**REFERENCES**


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